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Research Article

An Iterative Soft Bit Error Rate Estimation of Any Digital Communication Systems Using a Nonparametric Probability Density Function

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In general, performance of communication system receivers cannot be calculated analytically. The bit error rate (BER) is thus computed using the Monte Carlo (MC) simulation (Bit Error Counting). It is shown that if we wish to have reliable results with good precision, the total number of transmitted data must be conversely proportional to the product of the true BER by the relative error of estimate. Consequently, for small BERs, simulation results take excessively long computing time depending on the complexity of the receiver. In this paper, we suggest a new means of estimating the BER. This method is based on an estimation, in an iterative and nonparametric way, of the probability density function (pdf) of the soft decision of the received bit. We will show that the hard decision is not needed to compute the BER and the total number of transmitted data needed is very small compared to the classical MC simulation. Consequently, computing time is reduced drastically. Some theoretical results are also given to prove the convergence of this new method in the sense of mean square error (MSE) criterion. Simulation results of the suggested BER are given using a simple synchronous CDMA system.

1. Introduction

The famous Monte Carlo (MC) simulation technique is the most popular technique used for estimating bit error rate (BER) of digital communication systems. The MC method is used when we cannot analytically compute the performance of communication system receivers. Unfortunately, it is well known that the drawback of the MC method is its very high computational cost. If we are studying, for example, a channel with a BER equal to 10^{-6}, it is shown that if we hope to have a relative error estimation equal to 10^{-1}, the number of the incorrect received bits must be at least equal to 10^2 and then the total number of transmitted data must be at least equal to 10^8 (see [1]). Consequently, simulation results take excessively long computing time. In this paper, we suggest a new method to estimate the BER based on an estimation, in an iterative and nonparametric way, of the probability density function (pdf) of the soft decision of the received bit. In this case, the hard decision is not needed to compute the BER. The total number of transmitted data needed is very small compared to the classical MC simulation. Consequently, computing time is reduced drastically. The paper is organized as follows. In Section 2, a brief review of the MC simulation method is given. Section 3 shows how a pdf can be estimated in a parametric way. Section 4 gives some details about the new suggested iterative soft BER estimation. The convergence of this new method in the sense of Mean Square Error (MSE) criterion is discussed in Section 5. Simulation results are presented in Section 6. Finally, a brief summary of the results is given in Section 7.

2. Monte Carlo Simulation: a Brief Review

In this section, we will give a brief description of the MC simulation for any digital communication system. Let us consider any point to point system communication over any channel transmission (Gaussian, multipath fading, etc.) with or without channel coding using any transmission techniques (CDMA, MC-CDMA, TDMA, etc.). Let $(b_i)_{1\le i\le N} \in \{+1,-1\}$ a set of N independent transmitted bits. Let $(X_i)_{1\le i\le N}$ be the corresponding soft output at the receiver such as the decision is taken by using its sign: $\hat{b}_i = \operatorname{sgn}(X_i)$.

Let us introduce the following error function defined by a Bernoulli random variable:

$$\xi(b_i) = \begin{cases} 1 & \text{if } \hat{b}_i \neq b_i, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let p_e be the true BER at the output of the receiver. We have

$$p_e = Pr[\hat{b}_i \neq b_i] = Pr[\xi(b_i) = 1] = \mathbb{E}[\xi(b_i)], \tag{2}$$

where $\mathbb{E}(\cdot)$ is the mathematical expectation operator. The MC method estimates BER using the following average:

$$\hat{p}_e = \frac{1}{N}\sum_{i=1}^{N}\xi(b_i). \tag{3}$$

The estimator error is given by

$$e = p_e - \hat{p}_e = \frac{1}{N}\sum_{i=1}^{N}(p_e - \xi(b_i)). \tag{4}$$

The MC estimator is unbiased since $\mathbb{E}(e) = 0$ and its variance is given by (assuming that the errors are independent)

$$\sigma_e^2 = \mathbb{E}\left[(p_e - \hat{p}_e)^2\right] = \frac{p_e(1-p_e)}{N}. \tag{5}$$

Let ε be the relating error of the MC estimator which is given by

$$\varepsilon = \frac{\sigma_e}{\mathbb{E}(\hat{p}_e)} = \sqrt{\frac{1-p_e}{p_e N}}. \tag{6}$$

For small BER ($p_e \ll 1$), we have

$$\varepsilon \approx \frac{1}{\sqrt{p_e N}}. \tag{7}$$

Equation (7) gives the number of transmitted data needed for a given BER and for a desired precision ε:

$$N = \frac{1}{\varepsilon^2 p_e}. \tag{8}$$

It is clear from (8) that, for example, if we wish to study a channel with a BER equal to 10^{-7} with a desired precision of 10^{-1}, we must transmit at least 10^9 information bits. Consequently, simulation results take excessively long computing time depending on the complexity of the receiver. So, small BER values require large samples length N. That is why, in the following sections, we will suggest a new method to estimate the BER based on nonparametric pdf of soft output decision X.

3. Nonparametric Probability Density Function Estimation

Let $f_X(x)$ be the pdf of the soft output decision X at the receiver. Let us note that all the received soft output decision $(X_i)_{1\le i\le N}$ are random variables having the same pdf, $f_X(x)$. X_i is the corresponding soft output at the receiver such as the hard decision is taken by using its sign: $\hat{b}_i = \operatorname{sgn}(X_i)$. The $(b_i)_{1\le i\le N}$ are assumed to be independent and identically distributed with $P[b_i = \pm 1] = 1/2$. The BER is then given by

$$\begin{aligned} p_e &= P[\hat{b}_i \neq b_i], \\ &= P[(X>0),(b_i=-1)] + P[(X<0),(b_i=+1)], \\ &= P[X>0 \mid b_i=-1]P[b_i=-1] \\ &\quad + P[X<0 \mid b_i=+1]P[b_i=+1], \\ &= P[X>0 \mid b_i=-1], \\ &= P[X<0 \mid b_i=+1], \end{aligned} \tag{9}$$

then,

$$\begin{aligned} p_e &= \int_{-\infty}^{0} f_X^{b_i=+1}(x)dx \\ &= \int_{0}^{+\infty} f_X^{b_i=-1}(x)dx \\ &= \frac{1}{2}\int_{-\infty}^{0} f_X^{b_i=+1}(x)dx + \frac{1}{2}\int_{0}^{+\infty} f_X^{b_i=-1}(x)dx, \end{aligned} \tag{10}$$

where $f_X^{b_i=+1}(\cdot)$ (resp., $f_X^{b_i=-1}(\cdot)$) is the conditional pdf of X such as $b_i = +1$ (resp., $b_i = -1$).

Equation (10) clearly shows that an alternative method for estimating the BER is to transmit, for example, a sequence of N bits equal to $+1$, estimate the pdf of the soft output of the receiver and then calculate the BER by computing the appropriate integral given by (10).

However, in a practical situation, the nature of the pdf of the observed random variable X depends on both the type of receiver and the channel model; Gaussian function for a simple additive white Gaussian noise (AWGN) channel, a mixture of Gaussian functions for an AWGN CDMA receiver, or other distributions used for Rayleigh, Nakagami, or Rice fading channels. In the case of advanced receivers using iterative techniques or nonlinear filters such as turbo codes for multiple input multiple output (MIMO) systems [2], it is very difficult to find the right parametric model for the received distribution. That is why, for any communication systems, we suggest using nonparametric methods to estimate the pdf of the observed data, X. In fact, the most popular nonparametric pdf estimations are the Kernel method [3, 4] or the orthogonal series estimators such as the Fourier series [5]. Recent suggestions for methods can be found in [6, 7] with applications for shape classification and speech coding. In this paper, we will focus on the use of the Kernel method and its use for estimating the BER.

The Kernel estimator is defined as

$$\hat{f}_{X,N}(x) = \frac{1}{Nh_N}\sum_{i=1}^{N} K\left(\frac{x - X_i}{h_N}\right), \tag{11}$$

where $(X_i)_{1\leq i\leq N}$ are random variables having the same pdf, $f_X(x)$. X_i is the soft output at the receiver right before the hard decision. h_N is the smoothing parameter which depends on the length of the observed samples, N. $K(\cdot)$ is any pdf (called the kernel) assumed to be an even and regular (i.e., square integrated) function with unit variance and zero mean.

The choice of the smoothing parameter h_N is very important. It is shown in [6, 7] that if h_N tends towards 0 when N tends towards $+\infty$, the estimator $\hat{f}_{X,N}(x)$ is asymptotically unbiased (i.e., for all x, $\mathbb{E}[\hat{f}_{X,N}(x)] \to f_X(x)$). It is also shown that if $h_N \to 0$ and $Nh_N \to +\infty$ when $N \to +\infty$, then the MSE of the Kernel estimator tends to zero, that is, for all x:

$$\lim_{N\to+\infty} \mathbb{E}\left[\left(\hat{f}_{X,N}(x) - f_X(x)\right)^2\right] = 0. \tag{12}$$

Moreover, the optimal smoothing parameter h_N is computed in the minimum of the Integrated Mean Squared Error (IMSE) sense. An approximation of the IMSE is given by the following formula: (see [8])

$$\text{IMSE} \approx \frac{M(K)}{Nh_N} + \frac{J(f_X)h_N^4}{4}, \tag{13}$$

where $M(K) = \int_{-\infty}^{+\infty} K^2(x)dx$, $J(f_X) = \int_{-\infty}^{+\infty}(f_X''(x))^2dx$ and $f_X''(x)$ is the second derivative of the pdf $f_X(x)$. The optimal smoothing value, h_N^*, is then given by minimising the IMSE. We then obtain

$$h_N^* = N^{-1/5}(J(f_X))^{-1/5}(M(K))^{+1/5}. \tag{14}$$

Equation (14) shows that we must compute $J(f_X)$ which unfortunately depends on the unknown pdf, f_X. In the rest of this paper, we suggest the use of the most popular Gaussian kernel: $K(x) = (1/\sqrt{2\pi})\exp(-x^2/2)$. In this case, using (11), we have (proof is given in Appendix A)

$$J(\hat{f}_{X,N}) = \frac{1}{N^2h_N^5\sqrt{2}}\sum_{i=1}^{N}\sum_{j=1}^{N} K\left(\frac{X_i - X_j}{\sqrt{2}h_N}\right)\left[\left(\frac{X_i - X_j}{2h_N}\right)^4 + \frac{3}{4}\right]. \tag{15}$$

Let us note that we can easily show that for a zero mean and unit variance Gaussian kernel, we have

$$M(K) = \int_{-\infty}^{+\infty} K^2(x)dx = \frac{1}{2\sqrt{\pi}}. \tag{16}$$

4. Soft BER Estimation

To find the optimal smoothing parameter h_N^*, we must resolve (14) using at the same time (15) and (16). Direct resolution seems to be very difficult. That is why we suggest resolving this equation in an iterative way; we begin by an initial value of h_N ($h_N^{(0)} = 1/N^{1/5}$), then, for each iteration k: compute $J(\hat{f}_{X,N})$ using (15) with the previous $h_N^{(k-1)}$ and then compute the new value of $h_N^{(k)}$ by using (14). Once the optimal smoothing parameter is calculated, the pdf $\hat{f}_{X,N}(x)$, if needed, can be estimated by using (11). To estimate the BER of our system, we must evaluate the expression of (10): $\hat{p}_e = \int_{-\infty}^{0}\hat{f}_{X,N}(x)dx$. We can show that for the chosen Gaussian kernel, a soft BER estimation can be given by the following expression (see proof in Appendix B):

$$\hat{p}_{e,N} = \frac{1}{N}\sum_{i=1}^{N} Q\left(\frac{X_i}{h_N}\right), \tag{17}$$

where $Q(\cdot)$ denotes the complementary unit cumulative Gaussian distribution, that is, $Q(x) = \int_x^{+\infty}(1/\sqrt{2\pi})\exp(-t^2/2)dt$. The erfc function can also be used as follows: $Q(x) = 1/2\ \text{erfc}(x/\sqrt{2})$.

Let us now summarize the new suggested algorithm which estimates the soft BER of any communication system:

Soft BER algorithm: Let $(X_i)_{1\leq i\leq N}$ be the received soft output decision (corresponding to an N transmitted sequence bits equal to +1, so as the estimated pdf will be the conditional one of X such as $b = +1$).

(1) *Initialization*. $h_N^{(0)} = 1/N^{1/5}$.

(2) *For each iteration k*: ($k = 1, 2, \ldots$)

  (i) Compute $J(\hat{f}_{X,N}^{(k)})$ using $h_N^{(k-1)}$ (15).

  (ii) Compute $h_N^{(k)}$ using $J(\hat{f}_{X,N}^{(k)})$ and $M(K)$ ((14) and (16)).

  (iii) STOP iteration criterion: $|h_N^{(k)} - h_N^{(k-1)}| <$ threshold $\approx 10^{-3}$.

(3) *Soft BER computation*: (see (17)).

5. Some Theoretical Studies

In this section we shall give some theoretical studies. The following theorem will show that the suggested soft BER estimator is asymptotically unbiased. Proof of this theorem is given in Appendix C.

Theorem 5.1. *Assume that f_X is a second derivative pdf function, that $h_N \to 0$ as $N \to +\infty$. Then $\hat{p}_{e,N}$ is asymptotically unbiased, that is,*

$$\lim_{N\to+\infty} \mathbb{E}[\hat{p}_{e,N}] = p_e. \tag{18}$$

The following theorem shows that the variance of the suggested estimator also tends to zero. Proof of this theorem is given in Appendix D.

Theorem 5.2. *Assume that f_X is a second derivative pdf function, that $h_N \to 0$ as $N \to +\infty$. Then, the variance of $\hat{p}_{e,N}$ tends to zero as N tends to $+\infty$, that is,*

$$\lim_{N\to+\infty} \mathbb{E}\left[\left(\hat{p}_{e,N} - \mathbb{E}[\hat{p}_{e,N}]\right)^2\right] = 0. \tag{19}$$

Using Theorems 5.1 and 5.2, it is easy to show (see Appendix E) that the suggested estimator is pointwise consistent, that is, the MSE tends to zero as the number of samples N tends to $+\infty$. This result can be given by the following corollary.

Corollary 5.3. *Assume that f_X is a second derivative pdf function, that $h_N \to 0$ as $N \to +\infty$. Then, the MSE of $\widehat{p}_{e,N}$ tends to zero as N tends to $+\infty$, that is,*

$$\lim_{N \to +\infty} \mathbb{E}\left[\left(\widehat{p}_{e,N} - p_e\right)^2\right] = 0. \tag{20}$$

In the following, some remarks are given.

(1) *Asymptotic normality*: Using the central limit theorem, we can show that the sequence of BER estimator $\widehat{p}_{e,N} = (1/N)\sum_{i=1}^{N} Q(X_i/h_N)$ is asymptotically normal, that is,

$$\forall c \in \mathbb{R}, \lim_{N \to +\infty} P\left[\frac{\widehat{p}_{e,N} - \mathbb{E}[\widehat{p}_{e,N}]}{\sigma[\widehat{p}_{e,N}]} \leq c\right] = \int_{-\infty}^{c} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy. \tag{21}$$

(2) *Boostrap*: As $\widehat{f}_{X,N}(x)$ is constructed by the Kernel estimator (11) for a given observation $X_1, X_2, \ldots, X_N$, with kernel K and bandwidth h_N, then it is easy to find new independent realizations from this estimator. It is not necessary to explicitly compute $\widehat{f}_{X,N}(x)$ in the simulation procedure. New realizations Y can be drawn as follows:

(i) uniformly choose an index i with replacement from the set $\{1, \ldots, N\}$;

(ii) generate a random variable ε having K as a pdf;

(iii) Set $Y = X_i + \varepsilon h_N^*$.

These new realizations can be used to improve the accuracy of the estimator and therefore reduce the variance of the estimator.

6. Simulation Results

Let us consider a simple example in order to verify that our suggested BER estimator works well. In this section, we shall consider a synchronous CDMA system with K users employing normalized spreading codes $\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_k} \in \{-1/\sqrt{\mathrm{SF}}, +1/\sqrt{\mathrm{SF}}\}^{\mathrm{SF}}$ of length SF chips, through an AWGN channel using binary phase-shift keying (BPSK), where SF is the spreading factor. The received signal is the superposition of the data signals of K users given by

$$\mathbf{r} = \sum_{k=1}^{K} A_k b_k \mathbf{s_k} + \mathbf{n}, \tag{22}$$

where,

$\mathbf{r} \in \mathbb{R}^{\mathrm{SF}}$ is the received signal (SF = spreading factor);

$\mathbf{s_k} \in \{+1/\sqrt{\mathrm{SF}}, -1/\sqrt{\mathrm{SF}}\}^{\mathrm{SF}}$ is the spreading code for the kth user;

$b_k \in \{+1, -1\}$ is the transmitted binary information symbol of the kth user;

A_k is the received amplitude of the kth user;

$\mathbf{n} \in \mathbb{R}^{\mathrm{SF}}$ is an additive white Gaussian noise with zero mean and a covariance matrix equal to $\sigma^2 I_{\mathrm{SF}}$, ($\mathbf{n} \sim \mathcal{N}(0, \sigma^2 I_{\mathrm{SF}})$).

It is seen [9] that a sufficient statistic for demodulating the data bits of the K users is given by the K-vector $\mathbf{y}$ whose kth component is the output of a filter matched to $\mathbf{s_k}$, that is,

$$y_k = \mathbf{s_k}^{\top} \mathbf{r}, \quad k = 1, \ldots, K. \tag{23}$$

Using (22) and (23), we can show that the output of the kth matched filter is given by

$$y_k = A_k b_k + \sum_{j \neq k} A_j b_j \rho_{j,k} + \widetilde{n}_k, \tag{24}$$

where $\rho_{j,k}$ is the normalized cross-correlation between the spreading codes $\mathbf{s_j}$ and $\mathbf{s_k}$, $\widetilde{n}_k$ is the output additive Gaussian noise ($\widetilde{n}_k \sim \mathcal{N}(0, \sigma^2)$).

Note that the quantity (24) consists of three terms: the required bit information of the kth user, $A_k b_k$; a term $\sum_{j \neq k} A_j b_j \rho_{j,k}$ which is the multiple access interference (MAI) at the output of the matched filter due to the presence of other users sharing the same channel; and a term $\widetilde{n}_k$, due to the output of the background noise through the matched filter. Let us note that the additive noise, MAI $+\widetilde{n}_k$, at the output of the kth matched filter is a mixture of 2^{K-1} Gaussian distribution.

Several multiuser detection methods are given in [9]. Here, we shall focus on the conventional detector which is given by

$$\widehat{b}_k = \operatorname{sign}(y_k). \tag{25}$$

We can show, using (24), that the true bit error rate of the kth user for the conventional detector is given by the following formula:

$$\mathrm{BER}_k = \frac{1}{2^{K-1}} \sum_{b_{-k} \in \{\pm 1\}^{K-1}} Q\left(\frac{A_k - \sum_{j \neq k} A_j b_j \rho_{j,k}}{\sigma}\right), \tag{26}$$

where $b_{-k} = (b_1, b_2, \ldots, b_{k-1}, b_{k+1}, \ldots, b_K) \in \{-1, +1\}^{K-1}$.

For numerical results, we focus, for example, on $K = 2$ users with $\mathrm{SF} = 7$. The two spreading codes are chosen as $\mathbf{s_1} = (+1, +1, +1, +1, -1, -1, -1)/\sqrt{7}$ and $\mathbf{s_2} = (-1, -1, +1, +1, -1, -1, -1)/\sqrt{7}$. We have found that the cross-correlation value of these two codes is equal to $\rho_{1,2} = 0.4286$. Figure 1 (resp., 2) gives the conditional pdf such as $b_1 = +1$ of the output of matched filter for user $k = 1$ and for a SNR = 6 dB (resp., SNR = 10 dB).

Figure 3 gives performance of the conventional CDMA detector based on the true bit error rate (see (26)) compared with the method suggested in this paper and based on soft

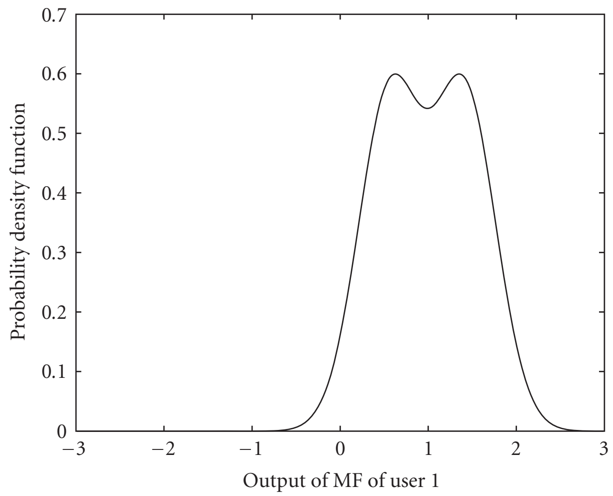

FIGURE 1: Conditional pdf such as $b_1 = +1$ of the output of matched filter for user $k = 1$ and for an SNR = 6 dB.

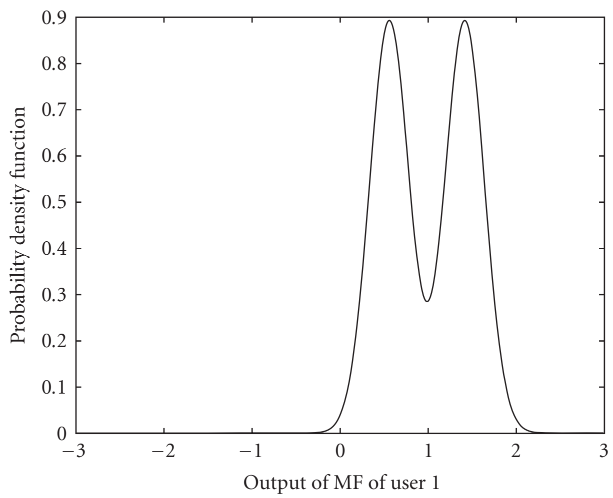

FIGURE 2: Conditional pdf such as $b_1 = +1$ of the output of matched filter for user $k = 1$ and for a SNR = 10 dB.

BER algorithm given in Section 4. For this last simulation, we have taken a database of length of $N = 1.000$ samples. The figure shows that for SNR = 10 dB, 1000 samples are sufficient to have a good precision of the bit error rate. For SNR = 10 dB, the true BER is equal to $3.0\,10^{-3}$, therefore, the MC simulation needs at least 30 000 samples for similar precision.

Other Receiver. In this section, instead of using a simple standard receiver, we shall consider a second example using an MMSE receiver which is an advanced technique using multiuser detection (see [9]). In this case, the output of the MMSE receiver is given by

$$\mathbf{z} = \mathbf{M}\mathbf{y}, \tag{27}$$

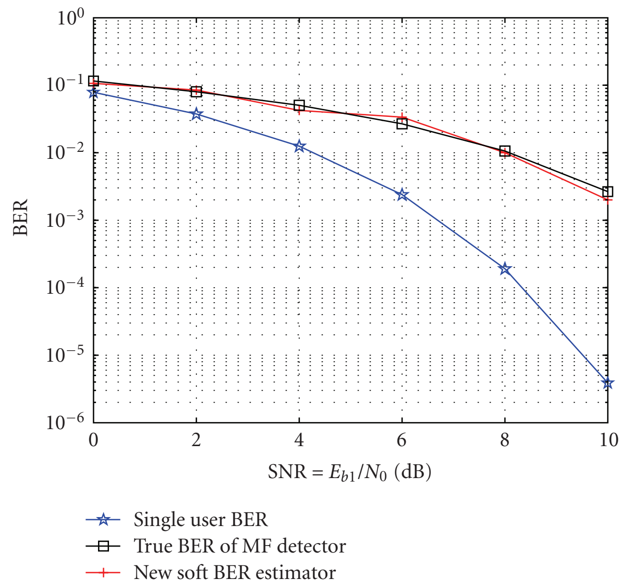

FIGURE 3: Soft BER algorithm and true BER comparison for synchronous CDMA system.

where $\mathbf{y} = [y_1, \ldots, y_K]^\top$ is the K-dimensional vector of matched filter outputs (y_k is given by (24)). The MMSE filter is given by

$$\mathbf{M} = \left[\mathbf{R} + \sigma^2\mathbf{A}^{-2}\right]^{-1}, \tag{28}$$

where $\mathbf{R}$ is the normalized cross-correlation matrix ($\mathbf{R}_{i,j} = \mathbf{s}_i^\top \mathbf{s}_j = \rho_{i,j}$), $\mathbf{A} = \operatorname{diag}\{A_1, \ldots, A_K\}$ and σ^2 is the variance of the additive white Gaussian noise.

The estimated bit for the kth user ($1 \le k \le K$) is then given by

$$\hat{b}_k = \operatorname{sign}(z_k) = \operatorname{sign}\left((\mathbf{M}\mathbf{y})_k\right). \tag{29}$$

We can show, using (27) and the fact that $\mathbf{y} = \mathbf{S}^\top \mathbf{r}$ (where $\mathbf{S} = [\mathbf{s_1}, \ldots, \mathbf{s_K}]$ is the $N \times K$ matrix of signature vectors), that the true bit error rate of the kth user for the MMSE receiver is given by the following formula:

$$\mathrm{BER}_k = \frac{1}{2^{K-1}} \sum_{b_{-k} \in \{\pm 1\}^{K-1}} Q\left(\frac{(\mathrm{MR})_{k,k}A_k - \sum_{j \ne k}(\mathrm{MR})_{k,j}A_j b_j}{\sigma\sqrt{(\mathrm{MRM})_{k,k}}}\right), \tag{30}$$

where $b_{-k} = (b_1, b_2, \ldots, b_{k-1}, b_{k+1}, \ldots, b_K) \in \{-1, +1\}^{K-1}$.

For numerical results, we focus on $K = 2$ users with the same spreading codes chosen for the first simulation (conventional detector). Let us note that the conditional pdf such as $b_1 = +1$ of the output of MMSE filter for each user is a mixture of 2^{K-1} Gaussian distribution.

Figure 4 gives performance of the MMSE receiver based on the true bit error rate (see (30)) compared with the method suggested in this paper and based on soft BER algorithm given in Section 4. For this last simulation, we have

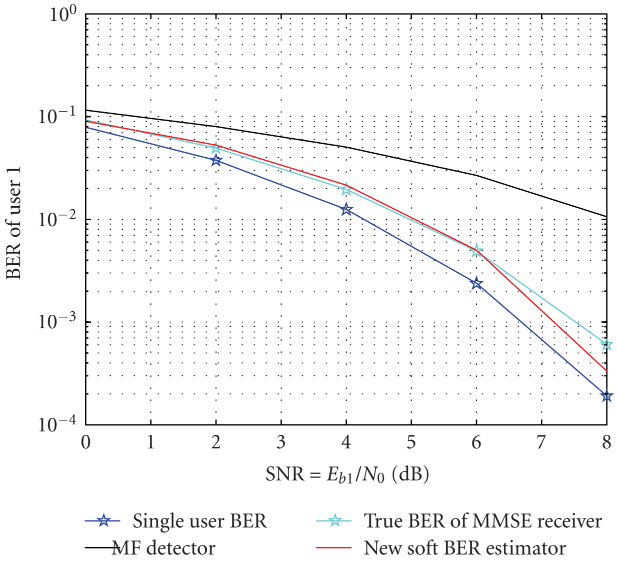

FIGURE 4: Soft BER algorithm and true BER comparison for synchronous MMSE-CDMA receiver.

taken a database of length of $N = 3.000$ samples. The figure shows that for SNR = 6 dB, 3000 samples are sufficient to have a good precision of the bit error rate. For such SNR, the true BER is equal to $5.0\,10^{-3}$, therefore, the MC simulation needs at least 50 000 samples for similar precision. Let us also note that for SNR = 8 dB, Figure 4 shows that the true BER is equal to $6.0\,10^{-4}$. In this case, the MC simulation needs at least 170 000 samples for a good precision. The soft BER estimation, by using only 3000 samples, gives a value of the BER with an error of 0.2 dB.

7. Conclusions

In this paper, we have suggested a new iterative soft bit error rate estimation for the study of any digital communication system performance. This method is based on the use of nonparametric pdf estimation of the soft decision of the received bit. Small length of transmitted data, compared to the MC method, is needed for the BER estimation. Convergence of this method in the MSE criterion has been proven. Some simulation results have been given in synchronous CDMA system case with both conventional detector and MMSE multiuser receiver.

Appendices

A. Proof of (15)

Proof. Using the definition of $J(f_X)$, we have

$$J(\hat{f}_{X,N}) = \int_{-\infty}^{+\infty} \left(f''_{X,N}(x)\right)^2 dx. \tag{A.1}$$

For Gaussian Kernel K, we have, $K''(x) = (x^2-1)K(x)$. Then, using (11), we have

$$\begin{aligned} J(\hat{f}_{X,N}) &= \frac{1}{N^2 h_N^6} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{-\infty}^{+\infty} \left[\left(\frac{x-X_i}{h_N}\right)^2 - 1\right] \\ &\quad \times \left[\left(\frac{x-X_j}{h_N}\right)^2 - 1\right] \\ &\quad \times K\left(\frac{x-X_i}{h_N}\right) K\left(\frac{x-X_j}{h_N}\right) dx, \\ &= \frac{1}{N^2 h_N^6} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{-\infty}^{+\infty} \left[\left(\frac{x-X_i}{h_N}\right)^2 - 1\right] \\ &\quad \times \left[\left(\frac{x-X_j}{h_N}\right)^2 - 1\right] \\ &\quad \times K\left(\frac{2x-(X_i+X_j)}{\sqrt{2}h_N}\right) K\left(\frac{X_i-X_j}{\sqrt{2}h_N}\right) dx. \end{aligned} \tag{A.2}$$

Let us use the following change of variable: $t = [2x-(X_i+X_j)]/\sqrt{2}h_N$ and let us note $a_{i,j} = (X_i - X_j)/2h_N$. we have

$$\begin{aligned} &\left[\left(\frac{x-X_j}{h_N}\right)^2 - 1\right]\left[\left(\frac{x-X_j}{h_N}\right)^2 - 1\right] \\ &\quad = \frac{t^4}{4} + t^2\left(2a_{i,j}^2 - 1\right) + \left(a_{i,j}^2 - 1\right)^2. \end{aligned} \tag{A.3}$$

Using both (A.2) and (A.3), we obtain

$$\begin{aligned} J(\hat{f}_{X,N}) &= \frac{1}{N^2 h_N^5 \sqrt{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} K(\sqrt{2}a_{i,j}) \\ &\quad \times \int_{-\infty}^{+\infty} \left[\frac{t^4}{4} + t^2\left(2a_{i,j}^2 - 1\right) + \left(a_{i,j}^2 - 1\right)^2\right] K(t)dt. \end{aligned} \tag{A.4}$$

For a zero mean and unit variance Gaussian Kernel, the second and fourth moment are, respectively, equal to 1 and 3, that is, $\int t^2 K(t)dt = 1$ and $\int t^4 K(t)dt = 3$. Therefore, (A.4) becomes

$$J(\hat{f}_{X,N}) = \frac{1}{N^2 h_N^5 \sqrt{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} K(\sqrt{2}a_{i,j}) \left(a_{i,j}^4 + \frac{3}{4}\right). \tag{A.5}$$

□

B. Proof of (17)

Proof. We must evaluate the expression of (10) in the case where $\hat{f}_{X,N}$ is estimated by Kernel method (see (11)). Then,

$$\begin{aligned} \hat{p}_{e,N} &= \int_{-\infty}^{0} \hat{f}_{X,N}(x)dx, \\ &= \int_{-\infty}^{0} \frac{1}{Nh_N} \sum_{i=1}^{N} K\left(\frac{x-X_i}{h_N}\right) dx. \end{aligned} \tag{B.1}$$

By using the following change of variable, $t = (x - X_i)/h_N$, we have

$$\begin{aligned}\hat{p}_{e,N} &= \int_{-\infty}^{0} \frac{1}{Nh_N}\sum_{i=1}^{N} K\left(\frac{x - X_i}{h_N}\right)dx \\ &= \sum_{i=1}^{N}\int_{-\infty}^{(-X_i/h_N)} \frac{1}{N}K(t)dt \\ &= \frac{1}{N}\sum_{i=1}^{N}\int_{-\infty}^{(-X_i/h_N)} \frac{1}{\sqrt{2\pi}}e^{-(t^2/2)}dt \\ &= \frac{1}{N}\sum_{i=1}^{N}\int_{(X_i/h_N)}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-(t^2/2)}dt \\ &= \frac{1}{N}\sum_{i=1}^{N} Q\left(\frac{X_i}{h_N}\right).\end{aligned} \tag{B.2}$$

□

C. Proof of Theorem 5.1

Proof. Let us first recall that the true BER is given by

$$p_e = \int_{-\infty}^{0} f_X(x)dx. \tag{C.1}$$

The suggested soft BER estimator is given by

$$\hat{p}_{e,N} = \int_{-\infty}^{0} \frac{1}{Nh_N}\sum_{i=1}^{N} K\left(\frac{x - X_i}{h_N}\right). \tag{C.2}$$

Then,

$$\begin{aligned}\mathbb{E}[\hat{p}_{e,N}] &= \int_{-\infty}^{0} \frac{1}{Nh_N}\sum_{i=1}^{N} \mathbb{E}\left[K\left(\frac{x - X_i}{h_N}\right)\right]dx \\ &= \int_{-\infty}^{0} \frac{1}{Nh_N}N\mathbb{E}\left[K\left(\frac{x - X_1}{h_N}\right)\right]dx \\ &= \int_{-\infty}^{0} \frac{1}{h_N}\left(\int_{-\infty}^{+\infty} K\left(\frac{x - u}{h_N}\right)f_X(u)du\right)dx.\end{aligned} \tag{C.3}$$

Using the following change of variable $t = (x - u)/h_N$, we have

$$\begin{aligned}\mathbb{E}[\hat{p}_{e,N}] &= \int_{-\infty}^{0} \frac{1}{h_N}\left(\int_{-\infty}^{+\infty} K(t)f_X(x - h_N t)dt\right)h_N dx \\ &= \int_{-\infty}^{0}\left(\int_{-\infty}^{+\infty} K(t)f_X(x - h_N t)dt\right)dx.\end{aligned} \tag{C.4}$$

As f_X is assumed to be second derivative pdf function, we can use Taylor series expansion of f_X as follows:

$$f_X(x - h_N t) = f_X(x) - h_N t f_X'(x) + \frac{h_N^2 t^2}{2}f_X''(x) + O(h_N^3 t^3). \tag{C.5}$$

Then, from (C.4), we have

$$\begin{aligned}\mathbb{E}[\hat{p}_{e,N}] &= \int_{-\infty}^{0}\Bigg(\int_{-\infty}^{+\infty} K(t)\Big[f_X(x) - h_N t f_X'(x) \\ &\quad + \frac{h_N^2 t^2}{2}f_X''(x) + O\left(h_N^3 t^3\right)\Big]dt\Bigg)dx \\ &= \int_{-\infty}^{0}\Bigg(f_X(x)\int_{-\infty}^{+\infty} K(t)dt - f_X'(x)h_N\int_{-\infty}^{+\infty} tK(t)dt \\ &\quad + \frac{h_N^2}{2}f_X''(x)\int_{-\infty}^{+\infty} t^2K(t)dt\Bigg)dx + O(h_N^3).\end{aligned} \tag{C.6}$$

As K is a zero mean and unit variance Gaussian Kernel, (C.6) becomes

$$\mathbb{E}[\hat{p}_{e,N}] = \int_{-\infty}^{0} f_X(x)dx + \frac{h_N^2}{2}f_X'(0) + O(h_N^3). \tag{C.7}$$

As $h_N \to 0$ when $N \to +\infty$, then

$$\lim_{N\to+\infty} \mathbb{E}[\hat{p}_{e,N}] = \int_{-\infty}^{0} f_X(x)dx = p_e. \tag{C.8}$$

□

D. Proof of Theorem 5.2

Proof. Let us first recall that the suggested soft BER estimator is given by

$$\hat{p}_{e,N} = \int_{-\infty}^{0} \frac{1}{Nh_N}\sum_{i=1}^{N} K\left(\frac{x - X_i}{h_N}\right), \tag{D.1}$$

then, the variance of this estimator can be computed as

$$\begin{aligned}\text{Var}\,[\hat{p}_{e,N}] &= \text{Var}\left[\int_{-\infty}^{0} \frac{1}{Nh_N}\sum_{i=1}^{N} K\left(\frac{x - X_i}{h_N}\right)\right]dx \\ &= \frac{1}{N^2h_N^2}N\ \text{Var}\left[\int_{-\infty}^{0} K\left(\frac{x - X_1}{h_N}\right)\right]dx \\ &= \frac{1}{Nh_N^2}\text{Var}(A),\end{aligned} \tag{D.2}$$

where A is given by

$$A = \int_{-\infty}^{0} K\left(\frac{x - X_1}{h_N}\right)dx. \tag{D.3}$$

Let us remark that from (D.1), we have

$$\mathbb{E}[A] = h_N\ \mathbb{E}[\hat{p}_{e,N}], \tag{D.4}$$

and then, using (C.7), we have

$$\mathbb{E}[A] = h_N p_e + \frac{h_N^3}{2}f_X'(0) + h_N O\left(h_N^3\right). \tag{D.5}$$

Now, to determine the analytical expression of (D.2), we must calculate $\mathbb{E}[A^2]$. Using (D.3), we have

$$\mathbb{E}[A^2] = \mathbb{E}\left[\int_{-\infty}^{0} K\left(\frac{x-X_1}{h_N}\right)dx \int_{-\infty}^{0} K\left(\frac{y-X_1}{h_N}\right)dy\right]. \tag{D.6}$$

We can easily show that for the chosen Gaussian kernel, we have

$$K\left(\frac{x-X_1}{h_N}\right)K\left(\frac{y-X_1}{h_N}\right) = K\left(\frac{X_1-(x+y/2)}{h_N/\sqrt{2}}\right)K\left(\frac{x-y}{\sqrt{2}h_N}\right). \tag{D.7}$$

Using (D.6), (D.7), and the following change of variable, $(v, w) = ((x + y/2), x - y)$, we have (using the fact that $K(\cdot)$ is a pdf and then $\int_{\mathbb{R}} K(w)dw = 1$)

$$\begin{aligned}
\mathbb{E}[A^2] &= \mathbb{E}\left[\int_{-\infty}^{0}\int_{-\infty}^{0} K\left(\frac{X_1-(x+y/2)}{h_N/\sqrt{2}}\right)K\left(\frac{x-y}{\sqrt{2}h_N}\right)dx\,dy\right] \\
&= \mathbb{E}\left[\int_{w=-\infty}^{+\infty}\int_{v=-\infty}^{0} K\left(\frac{X_1-v}{h_N/\sqrt{2}}\right)K\left(\frac{w}{\sqrt{2}h_N}\right)dv\,dw\right] \\
&= \mathbb{E}\left[\sqrt{2}h_N\int_{-\infty}^{0} K\left(\frac{X_1-v}{h_N/\sqrt{2}}\right)dv\right].
\end{aligned} \tag{D.8}$$

Then

$$\mathbb{E}[A^2] = \sqrt{2}h_N\int_{u\in\mathbb{R}}\left(\int_{-\infty}^{0} K\left(\frac{u-x}{h_N/\sqrt{2}}\right)dx\right)f_X(u)du, \tag{D.9}$$

using the following change of variable, $t = (u - x)/(h_N/\sqrt{2})$, we have

$$\mathbb{E}[A^2] = h_N^2\int_{t\in\mathbb{R}}\int_{-\infty}^{0} K(t)f_X\left(x+\frac{th}{\sqrt{2}}\right)dt\,dx. \tag{D.10}$$

As f_X is assumed to be a second derivative pdf, we can use Taylor series expansion of f_X as follows

$$f_X\left(x+\frac{th_N}{\sqrt{2}}\right) = f_X(x)+\frac{th_N}{\sqrt{2}}f_X'(x)+\frac{t^2h_N^2}{4}f_X''(x)+O\left(t^3h_N^3\right). \tag{D.11}$$

Then, from (D.10) and (D.11), we have (using the fact that K is a zero mean and unit variance Gaussian kernel)

$$\begin{aligned}
\mathbb{E}[A^2] &= h_N^2\int_{t\in\mathbb{R}}\int_{-\infty}^{0} K(t)f_X(x)+\frac{tK(t)h_N}{\sqrt{2}}f_X'(x) \\
&\quad+\frac{t^2K(t)h_N^2}{4}f_X''(x)dt\,dx \\
&= h_N^2\left[\int_{-\infty}^{0} f_X(x)dx+\frac{h_N^2}{4}f_X'(0)\right]+O\left(h_N^5\right) \\
&= h_N^2\left[p_e+\frac{h_N^2}{4}f_X'(0)\right]+O\left(h_N^5\right).
\end{aligned} \tag{D.12}$$

Using (D.2), (D.5), and (D.12), we obtain

$$\begin{aligned}
&\mathrm{Var}[\hat{p}_{e,N}] \\
&\quad= \mathbb{E}[A^2]-(\mathbb{E}[A])^2 \\
&\quad= \frac{1}{Nh_N^2}\left[h_N^2\left[p_e+\frac{h_N^2}{4}f_X'(0)\right]-\left(h_Np_e+\frac{h_N^3}{2}f_X'(0)\right)^2\right].
\end{aligned} \tag{D.13}$$

Then,

$$\begin{aligned}
\mathrm{Var}[\hat{p}_{e,N}] &= \frac{p_e(1-p_e)}{N}+\frac{h_N^2}{N}f_X'(0)\left(\frac{1}{4}-p_e\right) \\
&\quad-\frac{h_N^4}{4N}\left(f_X'(0)\right)^2+\frac{1}{N}O(h_N^5).
\end{aligned} \tag{D.14}$$

As $h_N \to 0$ as $N \to +\infty$, therefore

$$\lim_{N\to+\infty}\mathrm{Var}[\hat{p}_{e,N}] = 0. \tag{D.15}$$

□

E. Proof of Corollary 5.3

Proof. We have,

$$\begin{aligned}
&\mathbb{E}\left[(\hat{p}_{e,N}-p_e)^2\right] \\
&\quad= \mathbb{E}\left[(\hat{p}_{e,N}-\mathbb{E}[\hat{p}_{e,N}]+\mathbb{E}[\hat{p}_{e,N}]-p_e)^2\right] \\
&\quad= \mathbb{E}\left[(\hat{p}_{e,N}-\mathbb{E}[\hat{p}_{e,N}])^2\right]+(\mathbb{E}[\hat{p}_{e,N}]-p_e)^2 \\
&\qquad+2\mathbb{E}[(\hat{p}_{e,N}-\mathbb{E}[\hat{p}_{e,N}])(\mathbb{E}[\hat{p}_{e,N}]-p_e)].
\end{aligned} \tag{E.1}$$

By developing the expression $\mathbb{E}[(\hat{p}_{e,N}-\mathbb{E}[\hat{p}_{e,N}])(\mathbb{E}[\hat{p}_{e,N}]-p_e)]$, it is easy to show that its value is equal to zero. Then, we have

$$\begin{aligned}
\mathbb{E}\left[(\hat{p}_{e,N}-p_e)^2\right] &= \mathbb{E}\left[(\hat{p}_{e,N}-\mathbb{E}[\hat{p}_{e,N}])^2\right] \\
&\quad+(\mathbb{E}[\hat{p}_{e,N}]-p_e)^2.
\end{aligned} \tag{E.2}$$

As $\hat{p}_{e,N}$ is asymptotically unbiased ($\mathbb{E}[\hat{p}_{e,N}]-p_e \to 0$ as $N \to +\infty$, see Theorem 5.1) and the variance of $\hat{p}_{e,N}$ tends to 0 as $N \to +\infty$ (see Theorem 5.2), then

$$\lim_{N\to+\infty}\mathbb{E}\left[(\hat{p}_{e,N}-p_e)^2\right] = 0. \tag{E.3}$$

This means that $\hat{p}_{e,N}$ is pointwise consistent. □

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