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Research Article

Distributed Transmit Beamforming without Phase Feedback

Chen Wang, Qinye Yin, Jingjing Zhang, Bo Hao, and Wei Li

Ministry of Education Key Laboratory for Intelligent Networks and Network Security, Xi'an Jiaotong University, 28 West Xianning Road, Xi'an, Shaanxi 710049, China

Correspondence should be addressed to Chen Wang, wangchen05@gmail.com

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Phase feedback and adjustment between wireless nodes greatly reduce the power efficiency of distributed beamforming. In this paper, we propose a distributed transmit beamforming method without any phase feedback between nodes. The concept of our approach is to have the received signals retrace their ways, so that the phase offset of the forward path compensates that of the backward path; as a result, signals from different nodes in-phase combine at the destination. Therefore, the received power or the communication range is increased. In order to implement the concept of "retracement", we also propose a transceiver prototype which is based on the Direct Digital Synthesis technique. Experimental and simulation results validate the effectiveness of our approach.

1. Introduction

Long-range communication is a typical issue in Wireless Sensor Networks (WSNs). Direct transmission from a sensor node to a distant destination node requires high transmit power, which is not feasible for WSNs because of the size and power supply constrains on wireless sensor nodes. Therefore, it turns to the multihop fashion when the destination is located beyond one-hop coverage. However, in some applications, there is no relay node between the source and destination. For example, an Unmanned Arial Vehicle (UAV) collects sensor information from a cluster of nodes deployed in a certain area as shown in Figure 1. Due to some safety considerations the UAV cannot fly low enough to be within the single node coverage. Therefore, cooperation among the nodes becomes one possible approach. However, the popular distributed space-time code [1] and space-frequency code approaches [2] are mainly focused on the enhancement of reliability but not this kind of large-scale fading occurring over long distances. Another approach exploiting multiple antennas is the smart antenna system (SAS), which reallocates the energy in space through the coherent combination of radio frequency (RF) signals and forms a beam at the destination. As a result, the communication range increases without any power enhancement on each antenna. The concept of SAS can be utilized in WSNs where coherent signals transmitting from multiple nodes constructively combine at the destination so that the received power is enhanced [3, 4]. For the case of N collaborative nodes, the received power gain is N^2 which means an N-times larger communication range for free space propagation.

Although distributed beamforming promises many advantages, it faces many practical challenges too. The first challenge is the frequency synchronization between nodes [5]. The traditional SAS does not suffer from this problem because all antennas share a common local oscillator, so the signals are coherent in nature. However, in distributed beamforming system, nodes have their independent oscillators; so there are frequency offsets between them. In order to synchronize the frequency, one node (e.g., destination) broadcasts a high-power reference signal to each source node and source node adjusts its frequency of oscillator to that of [6–8]. Second, phase synchronization is also needed so that the signals are constructively combined at the destination, or they will cancel out each other. It should be noted that our phase synchronization means that signals have the same phase at the destination, not at the sources. The work in [9] divides the phase synchronization methods into closed-loop and open-loop approaches, which either synchronize the phases of nodes one by one [10] or adjust the phases of all nodes simultaneously [11, 12]. However, they all require the feed back of phase adjustment information from a certain node. The work in [13] proposes a scheme that does not require any phase precompensation. Instead, the destination node broadcasts a node selection vector to the pool of available source nodes to opportunistically select a subset of nodes whose transmitting signals combine in a quasi-in-phase manner at the destination. However, this node selection vector is also a feedback. Moreover, the energy on all the nodes cannot be fully exploited because it only selects a subset of available nodes. In one word, the feedback procedure decreases the efficiency of distributed beamforming.

This paper proposes a distributed transmit beamforming scheme which eliminates the inefficient feedback procedure. It utilizes the reciprocity of the signal propagation in space. By reversing the transmit sequence, the transmitting signals of the collaborative nodes "retrace their ways", and the phase shifts from the forward and backward path are automatically cancelled out so that they are in-phase combined at the destination. The collaborative nodes synchronize to the reference signal simultaneously and independently. The complexity of the network does not increase with the number of collaborative nodes, which fits the massively deployed WSNs.

The rest of the paper is organized as follows. Section 2 describes the system model for distributed beamforming. Then, we introduce our proposed beamforming method with discussion on the implementation aspect in Section 3. Section 4 simulates the influence of frequency and phase synchronization errors on performance. A hardware experiment is described in Section 5, which demonstrates the validity of distributed beamforming. Finally, the conclusions are given in Section 6.

2. Models and Main Assumptions

2.1. Network Structure. Nodes in the system are classified into two kinds: source node and destination node. A source node is typically characterized by low-cost, small size, resource-constrained, power limited, and so forth, while a destination node usually has more resources, adequate power supply, and higher transmission power so that all the source nodes could receive its signal. In WSNs, the wireless sensor nodes (source nodes) usually need to send their gathered messages to the data sink node; so the destination is also referred to as the sink node. Source collects data and transmits them back to the destination for further data analysis. However, a single source node constrained by node transmit power cannot directly send the information back to the destination node by single-hop. In this case, the source nodes perform distributed beamforming at the distant destination, making the signals in-phase combined there so that information is delivered by just one hop.

2.2. Signal Model. We adopt the free-space signal propagation model with assumption of line-of-sight (LOS) channels

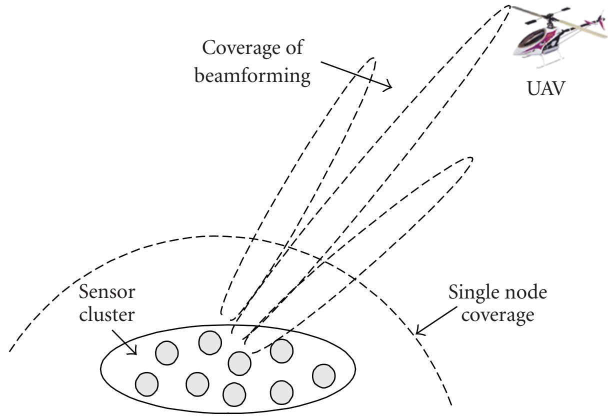

Figure 1: Single node cannot communicate with UAV directly.

to simplify our discussion. We first consider the change of the signal between a transmitter and a receiver during the communication process. The transmitting signal is

$$s(t) = e^{j\omega t}. \quad (1)$$

The received signal is the attenuated transmitted signal with time delay τ, that is,

$$r(t) = gs(t - \tau), \quad (2)$$

where g is the amplitude gain in free-space propagation, $g = \lambda/4\pi d$, λ is the signal wavelength, d is the distance between the nodes, τ is the propagation delay, and $\tau = d/c$, c is the speed of light. For a narrow band signal, say sinusoidal signal, the time delay represents phase shift, that is,

$$r(t) = ge^{-j\omega\tau}s(t). \quad (3)$$

Distributed beamforming at the destination node is to make the transmitting signals of multiple nodes have the same phase at the destination, that is, in-phase combination.

3. Beamforming without Phase Feedback

For a cluster of densely deployed sources, their gathered information is often highly correlated. So usually the data are firstly fused among the sources, and then sent back to the destination. Assume that before beamforming, all collaborative nodes have already obtained the common message $m(t)$ which is to be transmitted. When adopting time division duplex (TDD) mode, our approach can be divided into two time slots: synchronization time slot and beamforming time slot. In synchronization slot, the signal of each source node will be synchronized to the reference signal broadcasted by the destination node; in beamforming time slot, all the transmitting signals of source nodes are coherently combined at the destination node. Because there is no reference signal for sources to synchronize in beamforming slot, two slots repeat alternately as shown in Figure 2. Obviously, the synchronization time slot does reduce the communication efficiency. However, we believe

Synchronization	Beamforming	Synchronization	Beamforming	· · ·

Figure 2: Time slots structure for TDD mode.

that it is more important to successfully send the valuable message back to the sink node than to transmit more efficiently in this scenario.

3.1. Synchronization Time Slot. In this time slot, the destination node broadcasts a high-power reference signal, say sinusoidal signal:

$$s_d(t) = e^{j\omega t}. \tag{4}$$

According to (3), the signal received at the ith source after delay and attenuation is

$$r_i(t) = g_i e^{-j\omega\tau_i} e^{j\omega t}, \tag{5}$$

where g_i and τ_i are channel gain and propagation delay between the ith source and the destination, respectively.

Every source node continuously adjusts its local oscillator signal $s_i(t)$ within the specified time slot, making its frequency and phase synchronize to those of the received signal. That is:

$$s_i(t) = \frac{r_i(t)}{g_i}. \tag{6}$$

There are many ways for frequency and phase synchronization. Readers may refer to [14, 15], and so forth.

3.2. Beamforming Time Slot. When frequency and phase synchronization is completed, the adjusted local oscillator signal will be output in time reverse order in beamforming slot, that is,

$$s_i'(t) = s_i(-t) = \frac{r_i(-t)}{g_i}, \tag{7}$$

and the message $m(t)$ will be modulated to the carrier $s_i'(t)$. After transmission, the signals from N sources combine at the destination node:

$$r_d(t) = \sum_{i=1}^{N} g_i m(t-\tau_i) s_i'(t-\tau_i). \tag{8}$$

Substituting (5) and (7) into (8), we obtain

$$r_d(t) = e^{-j\omega t} \sum_{i=1}^{N} \left(g_i m(t-\tau_i)\right). \tag{9}$$

From (9) we note that the phase shift between the source and the destination has been cancelled through the reverse transmission of signals. For the common message $m(t)$, due to the low data rate in sensor networks, we consider $m(t)$ as a narrowband signal compared with the carrier ω, then $m(t-\tau_i) \approx m(t)$. The received signal can be rewritten as

$$r_d(t) = m(t) e^{-j\omega t} \sum_{i=1}^{N} g_i. \tag{10}$$

So transmitted signals in-phase combine at the destination node, and the received amplitude is the summation of the amplitudes of N nodes.

Due to the long distance between the source node and the destination node, the channel gain g_i between each source and the destination can be regarded as the same, namely, $g_i = g$. Then the received signal power is

$$P_r(t) = |r_d(t)|^2 = \left|Ngm(t)\right|^2 = N^2 \left|gm(t)\right|^2. \tag{11}$$

Compared with a single-node case, there is an enhancement of $20\log_{10}N$ dB in received power.

Note that for non-LOS environment, as long as the backward propagation environment is consistent with the forward one, the above method can also result in in-phase combination at the destination. Theoretically, the scheme of the proposed method is feasible if the synchronization process can catch up with the change of wireless channels. As far as our following experiment concerns, we just carried out experiments in static environment.

3.3. Discussion on Design and Implementation of Source Node. One key point of the proposed method lies in the transmission of the time-reversed signal from source node. However, the local oscillator of the RF transceiver usually has a relatively high frequency. It is very difficult to control its frequency and phase directly, especially for the low-cost wireless sensor nodes. We adopt the direct-conversion architecture (zero-IF) transceiver which is popular in low-cost applications. Based on this architecture, we set another "oscillator" in the baseband to control frequency and phase indirectly so that the upconverted RF signal is coherent. For digital transceivers it can be implemented as in Figure 3.

The baseband oscillator can be implemented with a direct digital synthesizer (DDS) which continuously generates complex sinusoid signal whose frequency and phase can be precisely controlled by its phase increment and phase offset register, respectively. In the synchronization slot, the sources work in receive mode and downconvert the sinusoidal signal broadcast from the destination, sample and convert to digital signal with ADCs, and then send the baseband IQ signal to the synchronization module. By comparing the received signal with the output of the baseband oscillator, synchronization module calculates the frequency and phase offset and adjusts those of the baseband oscillator so as to make them synchronized [14, 15]. In this case, frequency synchronization is achieved by adjusting the phase increment of the DDS, while phase synchronization is achieved by adjusting the phase offset register of the DDS. The synchronization process can be done in iterative approach so as to obtain an accurate result by simple algorithm. Due to the limitation of the paper, we will describe the specific details of the implementation in our future papers. In beamforming slot, the sources switch to transmit mode with the baseband oscillator running continuously but in reverse sequence. To achieve this, the only change of the transceiver is the sign of phase increment, which turns the phase accumulator in the synchronization time slot into a phase decreaser in the beamforming time slot as demonstrated in Figure 4.

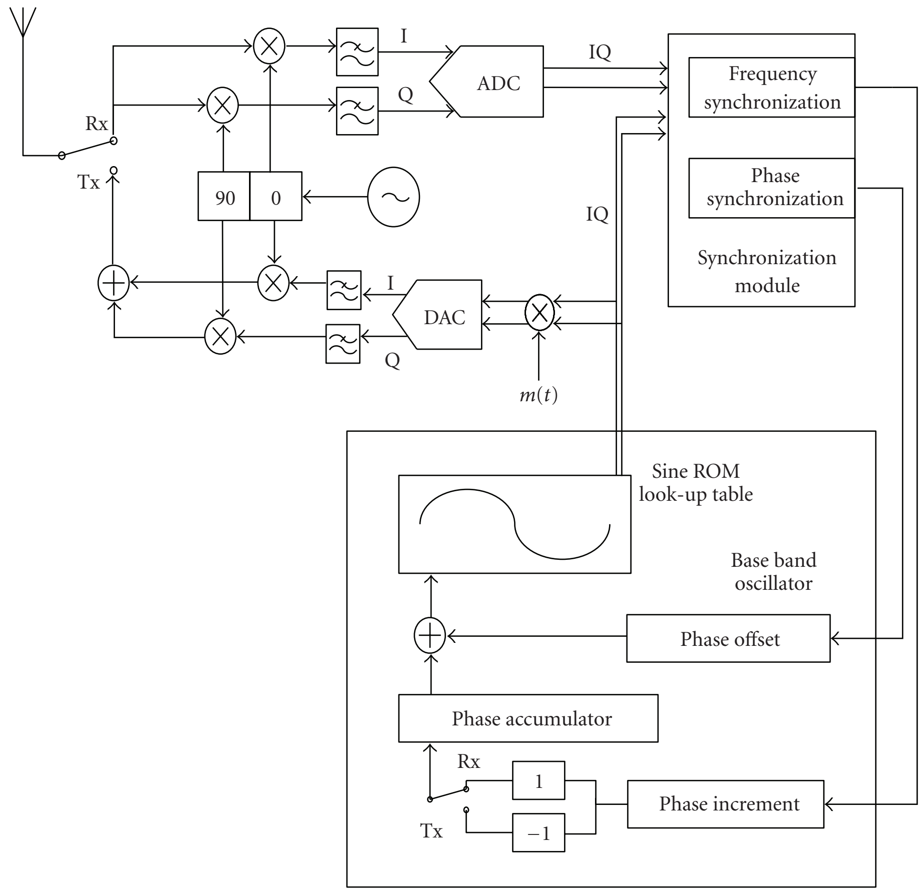

FIGURE 3: The transceiver uses a baseband oscillator to control the RF signal indirectly.

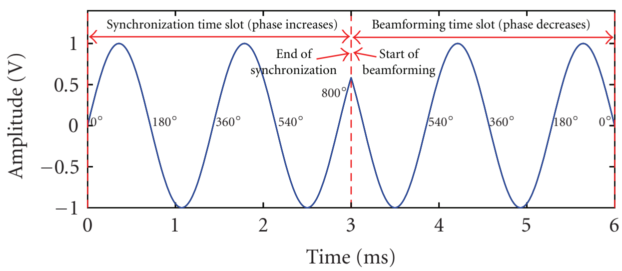

FIGURE 4: Output waveform of DDS in beamforming time slot is in reverse time sequence to that in synchronization time slot.

The message $m(t)$ is modulated by the reversed baseband oscillator, converted to analogue signal with DACs, and finally upconverted to RF signal.

4. Simulations

As aforementioned, during the synchronization slot, sources should synchronize frequency and phase in order to make the local signal have the same phase and frequency as that of the received signal. However, in reality, absolute frequency and phase coherence is impossible, because of the noise and the instability of the crystal oscillator, and so forth. Suppose that the transmit signal of the ith source has frequency error Δf_i and phase error $\theta_{i,e}$, then according to (10), the received signal at the destination is

$$r_d(t) = m(t)e^{-j\omega t}\sum_{i=1}^{N}\left(ge^{-j(2\pi\Delta f_i t+\theta_{i,e})}\right), \qquad (12)$$

which indicates that the amplitude of the received signal is no longer the sum of signals' amplitudes. The following simulation shows the effect of frequency and phase error on beamforming performance.

4.1. Effect of Phase Error on Performance. Suppose that there is only phase error but no frequency error, and then the power of the received signal at the destination node is

$$P_r(t) = \left|\sum_{i=1}^{N}\left(e^{-j\theta_{i,e}}\right)\right|^2 \left|gm(t)\right|^2. \qquad (13)$$

It shows that transmit signals do not completely in-phase combine at the destination node, which reduces the amplitude of the received signal.

To simplify our analysis, suppose that signal transmitted from each source has the same power at the destination node, that is, $gm(t) = 1$. Phase error is modeled as a zero-mean

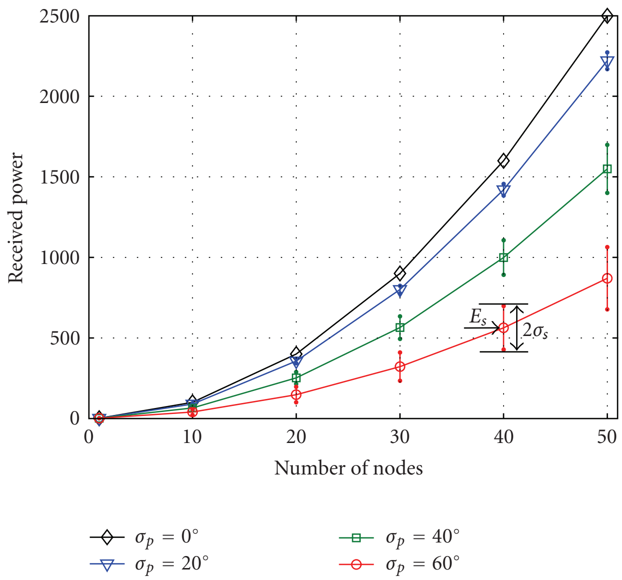

Figure 5: Relation between received power and the number of nodes under different phase errors.

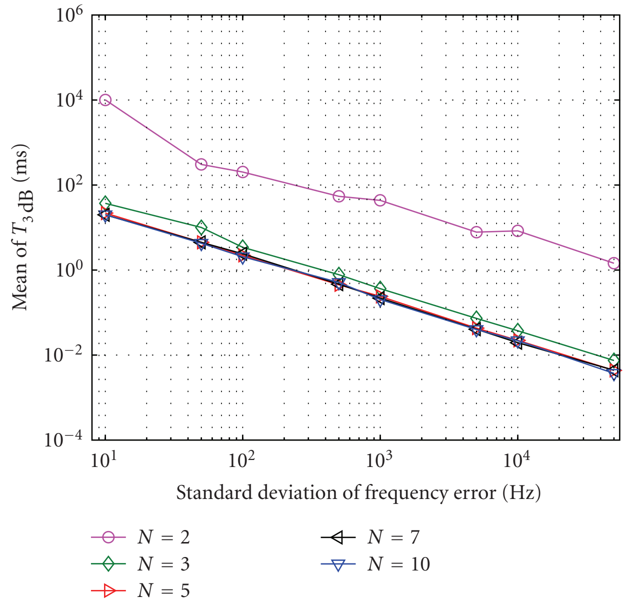

Figure 6: Relation between $T_{3\,\mathrm{dB}}$ and frequency synchronization error with different number of nodes.

Gaussian random variable $\theta_{i,e} \sim N(0, \sigma_p^2)$. Figure 5 illustrates the relationship between the number of collaborative nodes and the received power under different phase errors. The average (E_s) and the standard deviation (σ_s) of the received power are obtained after 5000 trials of simulation.

As is shown in Figure 5, the average received power increases with node number. However, it is not proportional to N^2, because of the existence of phase error. With the phase error increasing, the average of the received power falls while its variance rises. Nevertheless, we note that the beamforming is not sensitive to phase error; when the variance of phase error is 40 degree, the received power could still reach 60% of the ideal value.

4.2. Effect of Frequency Error on Performance. Because phase is the integration of frequency over time, a small frequency synchronization error can result in a large phase drift. Even though all the signals are synchronized at the beginning, they are not in-phase any more as time goes by, which causes the received power at the destination to fluctuate with time. The bigger frequency offset is, the faster the received power changes. Here we define the coherent collaborative time $T_{3\,\mathrm{dB}}$ as the time duration in which the received power drops from its peak to its half level (for two-node case the half power is 3 dB lower than its peak; so we denote it with subscript 3 dB):

$$T_{3\,\mathrm{dB}} = \min\left\{t \mid P_r(t) < \frac{P_{r,\max}}{2}\right\}. \quad (14)$$

It indicates the time duration in which distributed beamforming with frequency offset could perform effectively. Figure 6 shows the relation between $T_{3\,\mathrm{dB}}$ and frequency errors with different number of collaborative nodes. In the simulation, we suppose that the phases of all the sensors are synchronized at the beginning of the beamforming, namely, $\theta_{i,e} = 0$, for all i in (12), and frequency error is modeled as a zero-mean Gaussian random variable $\Delta f_i \sim N(0, \sigma_f^2)$.

According to Figure 6, $T_{3\,\mathrm{dB}}$ decreases as frequency offset increases. Besides, the number of nodes also affects $T_{3\,\mathrm{dB}}$. When the number of sensors is small ($N < 5$), $T_{3\,\mathrm{dB}}$ falls rapidly as N increases. $T_{3\,\mathrm{dB}}$ does not change greatly any more with N for $N > 7$. When the standard deviation of frequency errors among sensors is 100 Hz, $T_{3\,\mathrm{dB}}$ is millisecond order of magnitude, which satisfies the demands of most of the communication protocols (e.g., the longest packet duration is 4.256 ms for IEEE 802.15.4).

5. Hardware Experiment

5.1. Verification of Coherent Transmission Using Independent Wireless Nodes. The following experiment demonstrates the feasibility of the coherent superposition of two independent sensors. The complexity and time-variance of wireless channel will affect the accuracy of measurement result. In order to analyse with an accurate measurement, some devices such as the power combiner are used to simulate the superposition of the RF signal while excluding the impact of the time-varying wireless environment. The experiment setup is shown in Figure 7. The RF signal which is combined from two transmitted signals by a combiner is input to a spectrum analyzer. The spectrum analyzer is set to Zero-Span mode in order to observe the change of the combined signal's power versus time. By observing the change of the signal power, we could obtain the quantitative indicator of the power enhancement for coherent combination.

We set the two nodes to the same frequency and phase by manually adjusting the control words of DDS. In the experiment, each sensor has an independent local oscillator, and there is frequency offset between them. By

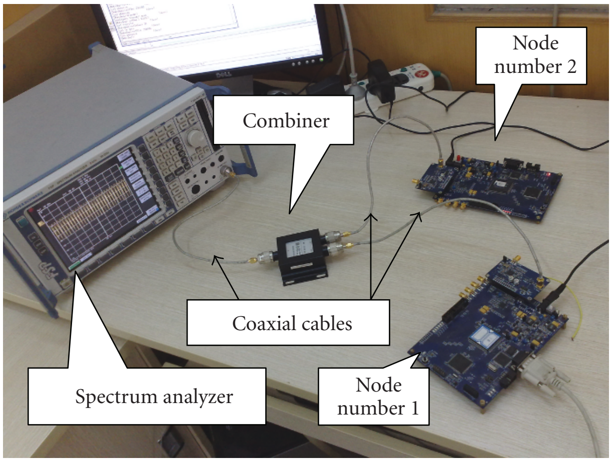

Figure 7: Experiment setup of the test platform.

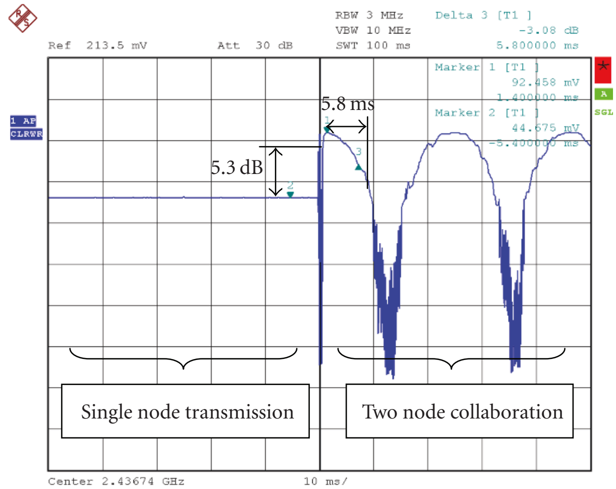

Figure 8: The measured received power for two-node collaboration.

adjusting DDS output frequency and phase of one sensor, the frequency and phase offsets are compensated manually. Figure 8 is the captured result from the spectrum analyzer. In the experiment, the signal amplitude of nodes no. 1 and no. 2 is 55.8 mV and 44.6 mV (Marker 2), respectively, and the maximum signal amplitude after combination is 92.4 mV (Marker 1); so the coherent collaboration achieves 92% of its ideal value. This equals to 5.3 dB power gain compared with single node transmission (the theoretical gain is 6 dB for two nodes transmission). From the experiment, we also see that the compensation at a certain time will be gradually invalidated with time because of the short-term instability of the oscillator's output frequency. However, this fluctuation in oscillator's frequency is slow and we just adopt some low cost oscillators whose frequency stability is above 25 ppm. Using TCXO (Temperature Compensated Crystal Oscillator with frequency stability typically lower than 1 ppm) will further slow this change. The measurement result shown in Figure 8 indicates that $T_{3\,\mathrm{dB}}$ is 5.8 ms (Delta Marker 3 with reference to Marker 1) and the period of the received signal's power is about 20 ms which equals to 50 Hz of the frequency error.

5.2. Experiment of the Proposed Method. The above section has already shown the feasibility of two independent nodes beamforming in real system. In this section, we implement the proposed method on hardware system and show its high beamforming efficiency from our experiment result. The test system platform is illustrated in Figure 9.

In order to simulate the amplitude attenuation in real space propagation, we add an attenuator of 40 dB at the radio front end of the destination node. The reference signal transmitted by the destination node is distributed to two source nodes through Combiner/Splitter 1 in synchronization time slot. In beamforming time slot, the transmitted signals from two sources are first combined by Combiner/Splitter 1, and then split to the destination node and the spectrum analyzer by Combiner/Splitter 2. The measurement results from the spectrum analyzer are shown in Table 1. In this experiment,

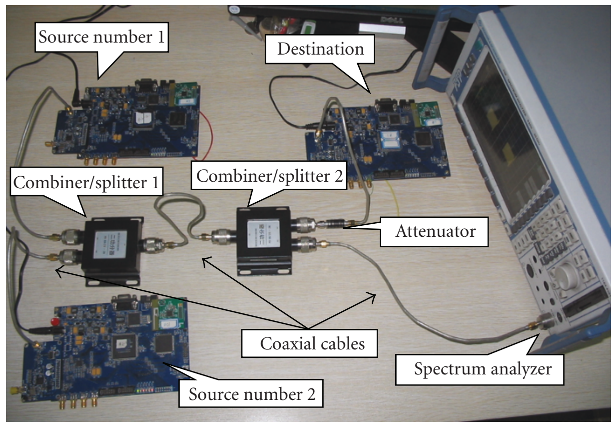

Figure 9: Experiment setup for the proposed method.

Table 1: Experiment results.

Transmitting node(s)	Received power of spectrum analyzer
Source no. 1	−5.11 dBm (124.16 mV)
Source no. 2	−3.92 dBm (142.39 mV)
Source no. 1 and no. 2	1.02 dBm (251.47 mV)

the time durations of synchronization and beamforming are set to around 3 ms.

Ideally, the maximum beamforming power of Source no. 1 and no. 2 can be 266.55 mV, while the actual measured result is 251.47 mV. The beamforming efficiency is 94.3% (equivalent beamforming gain is 5.5 dB with 6 dB for ideal situation) compared with the experiment result of beamforming efficiency 90.3% (5.1 dB beamforming gain) in [9].

We also provide some specific measured waveform charts and discuss the results briefly. First we consider the case in which only one source (Source no. 1) transmits the signal, and the time-domain power of the received signal is shown

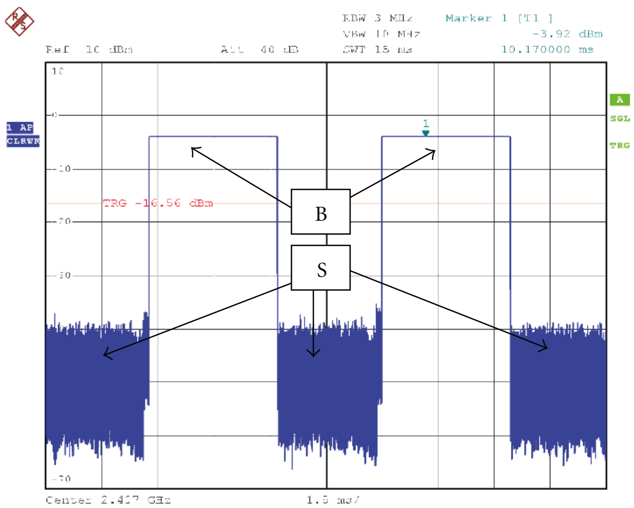

Figure 10: Only one source node is switched on. Note: that S represents the synchronization time slot during which sources are in receive mode and B represents the beamforming time slot.

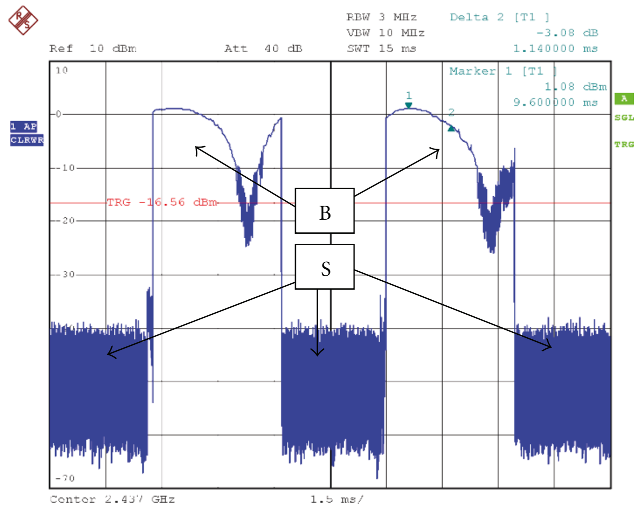

Figure 12: A bigger frequency difference of source nodes leads to a shorter beamforming coherent collaborative time.

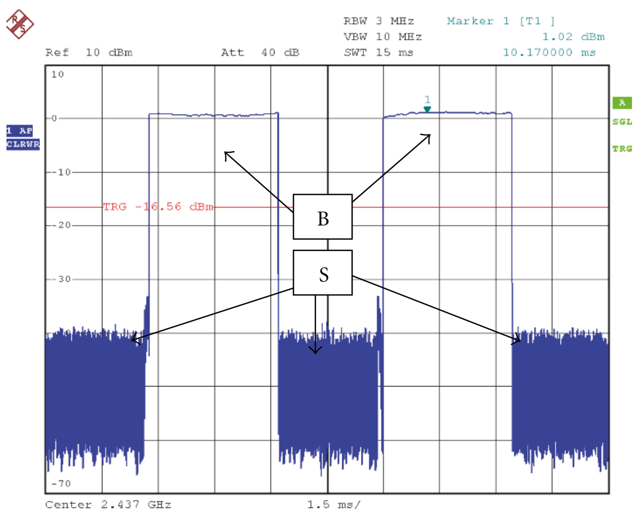

Figure 11: Received power waveform in time-domain when two sources beamform at the destination.

in Figure 10. A similar result is obtained when only Source no. 2 transmits, except with different amplitude.

When the two sources work at the same time, their signals combine in-phase at the destination according to the proposed beamforming technique. Figure 11 shows the received power of beamforming signal.

We can see from Figure 11 that the received power is relatively high during the entire beamforming slot, which indicates that the coherent collaborative time $T_{3\,\text{dB}}$ is longer than 3 ms. Because the two sources perform good frequency-phase synchronization, there is just a small frequency difference which has a very limited influence on beamforming result during the 3 ms. However, when we increase the noise or decrease the transmit power of the reference signal, the performance of the synchronization deteriorates, which results in a short coherent collaborative time as shown in Figure 12.

6. Conclusion

In this paper, we present a distributed transmit beamforming method, whose most distinct characteristic is to make the signals "retrace their ways to the destination". Consequently, it avoids the complicated feedback adjustment process. In addition, the simulation shows that an increase in the number of collaborative nodes, the phase error, or the frequency synchronization error results in decrease of the beamforming efficiency. Moreover, a transceiver reference prototype based on DDS is introduced, and some hardware experiments based on this architecture have been conducted. Experiment result shows the feasibility of coherent transmission among independent wireless nodes. In the case of two nodes beamforming using the proposed method, the received signal power achieves 92% of its ideal value, which is 5.3 dB higher compared with signal node transmission.

Acknowledgments

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