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Research Article

Stability Analysis of Hybrid ALOHA

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We perform stability analysis of a recently proposed MAC protocol, hybrid ALOHA, based on the multipacket reception (MPR) model. Hybrid ALOHA distinguishes from conventional slotted ALOHA by allowing conditional collision-free channel estimation and simultaneous transmissions, and hence improves the MPR capability of the system. The stability analysis of the two-user case ($N = 2$) has been conducted in our previous work. In this paper, we study the stability region of hybrid ALOHA for the general N-user ($N > 2$) system. By using the method of stochastic dominance and mathematical induction, we obtain the sufficient condition for the stability of hybrid ALOHA. As an illustration, we characterize the stability inner bounds for the $N = 3$ case. In this particular situation, the results are derived by solving a nonhomogeneous Riemann-Hilbert problem. Potentially, the mathematical tools used in this paper can be exploited for obtaining closed-form results in stability analysis of wireless networks.

1. Introduction

The study of interacting queueing systems has received enormous attention due to their importance in applications (e.g., multiple-access channel models and shared computer processor systems) as well as to their theoretical interest. However, the theoretical analysis of such systems has inherent difficulties due to the coupling of the queues (users). Assume there are a total of N users in the system. In [1], Fayolle and Iasnagorodski have displayed the mathematical difficulty of the analysis of two coupled users ($N = 2$). For systems with more than two users ($N > 2$), the analysis is even more arduous due to the higher dimensionality of the problem. Hence, the study of $N > 2$ systems becomes a challenging task and deserves more research attention.

Among various design tasks of queueing systems, a fundamental issue is the *stability*, which can be roughly understood as the ability of a system to keep the queue length in a bounded region. Extensive research on stability has been carried out for slotted ALOHA [2], which is probably the simplest system. A historic retrospection, however, reveals that in spite of its simple form, queueing theoretic analysis of ALOHA turns out to be truly difficult even under the so-called *collision model* [3]. Under this simplified model, a user is assumed to be successful in packet transmission if and only if there are no simultaneous transmissions from other users. In 1979, Tsybakov and Mikhailov initialized in [4] the stability analysis of finite-user slotted ALOHA and found the ergodicity conditions of the system, that is, the conditions under which the queues remain finite with probability 1. They found a simple bound for the stability region and also obtained the explicit stability region for the $N = 2$ users case. Szpankowski presented in [5] some improved bounds for the average queue lengths. Even tighter lower (inner) bounds for the stability region of the system with an arbitrary finite number of users were derived by Rao and Ephremides in [6], where a series of hypothetical auxiliary systems of queues that closely parallel the operation of the system of interest were constructed, and the inner bounds were obtained by means of stochastic dominance. The exact stability region of ALOHA for the finite-user case was obtained for a simple arrival process by Anantharam [7], yet the general results for arbitrary arrival distributions are still unknown. In [8], Szpankowski found the sufficient and necessary conditions for the stability of queues for a fixed transmission probability vector for the $N > 2$ case. However, the difficulty in computation of the stationary joint queue statistics makes it hard, if not impossible, to verify these conditions. Luo and Ephremides [9] introduced the concept of instability ranks in queues to obtain tight inner and outer bounds on the $N > 2$ case. Despite all the efforts, to date there is still no closed-form characterization of the stability region for the general finite-user case.

All the works discussed above were based on the collision model. A breakthrough was made by Naware et al. in [10], where the multipacket reception (MPR) model originally proposed by Ghez et al. [11, 12] was adopted for slotted ALOHA, enabling the feasibility of the successful simultaneous transmissions from different users. The stability region for $N = 2$ case under the MPR model was characterized and the result was extended to the *symmetric* $N > 2$ user system. In [13], Luo and Ephremides presented an interesting result about the equivalence between the stability region and the throughput region based on a conjectured sensitivity monotonicity property over "standard" MPR channels. The coincidence of the stability and throughput regions was further strengthened by Shrader and Ephremides [14] in considering a 2×2 broadcast network. However, these results relied on the validity of the conjectured property, which remained unproven and necessitated further research efforts.

Recently, based on the MPR model, we proposed a capacity-reaching random access protocol [15], named hybrid ALOHA. By allowing conditional collision-free channel estimation and simultaneous transmissions, hybrid ALOHA improves the MAC layer MPR capability and outperforms the traditional slotted ALOHA in terms of throughput, stability, and delay. The stability regions and delay bounds have been studied for the $N = 2$ case in [15]. In this paper, we further investigate the stability condition of the hybrid ALOHA system for the general $N > 2$ case. The results for the $N = 2$ case are used as the arguments of the mathematical induction for deriving the sufficient condition of the stability of the $N > 2$ system. By means of stochastic dominance, we characterize the stability inner bounds for the general N-user system. However, the explicit characterization of these bounds is nontrivial. As an illustration, we study the characterization of the stability inner bounds for the $N = 3$ case. Starting from a system functional equation, we show that characterizing such inner bounds reduces to solving a general Riemann-Hilbert problem. Potentially, the mathematical tools used in this paper can be applied for obtaining closed-form results in stability analysis, albeit they could be in very complicated forms.

The organization of the paper is as follows. In Section 2, we describe the system model and present the generalized hybrid ALOHA protocol. In Section 3, we derive the sufficient condition for stability of $N > 2$ systems. For the special case of $N = 3$, Section 4 derives the stability inner bound of the system. The problem is reduced to solving a Riemann-Hilbert boundary value problem. Section 5 concludes the article.

2. The Hybrid ALOHA Protocol

2.1. System Model. Consider a wireless network with a set $\mathcal{N}$ of users, $\mathcal{N} = \{1, 2, \ldots, N\}$, communicating with a common access point. Each user is equipped with an infinite buffer for storing arriving and backlogged packets. The packet arrival processes are assumed to be independent from user to user. The channel is slotted in time, with slot period larger than the packet length. When the buffer of the ith ($i \in \mathcal{N}$) user is nonempty, he/she transmits with probability p_i. Packets are assumed to be of equal size for all users and composed of two parts: the first part is the *training sequence* for channel estimation and the second part is the *information data*. The length of the training sequence is typically much smaller than that of the information data. The arrivals of the ith user are assumed to be independent and identically distributed (i.i.d) Bernoulli random variables from slot to slot, with the average number of arrivals being λ_i *packets per slot*.

We adopt the general MPR model in [10] where the multiuser physical layer is characterized by a set of conditional probabilities. For any subset $\mathcal{S} \subseteq \mathcal{N}$ of users transmitting in a slot, the *marginal* probability of successfully receiving packets from users in $\mathcal{R} \subseteq \mathcal{S}$, given that users in $\mathcal{S}$ transmit, is defined as

$$q_{\mathcal{R}|\mathcal{S}} = \sum_{\mathcal{U}:\mathcal{R}\subseteq\mathcal{U}\subseteq\mathcal{S}} q_{\mathcal{U},\mathcal{S}}, \tag{1}$$

where $q_{\mathcal{U},\mathcal{S}}$ is the conditional probability of reception defined as

$$q_{\mathcal{U},\mathcal{S}} = Pr\{\text{only packets from } \mathcal{U} \text{ are successfully received} \mid \text{users in } \mathcal{S} \text{ transmit}\}, \quad \mathcal{U} \subseteq \mathcal{S}. \tag{2}$$

In the two-user case, for example, $\mathcal{N} = \{1, 2\}$, for $i = 1, 2$,

$$\begin{aligned} q_{i,\{i\}} &= Pr\{\text{user } i \text{ is successful} \mid \text{only user } i \text{ transmits}\}, \\ q_{i,\{1,2\}} &= Pr\{\text{user } i \text{ is successful} \mid \text{both users transmit}\}, \\ q_{\{1,2\},\{1,2\}} &= Pr\{\text{both users are successful} \mid \text{both users transmit}\}, \end{aligned} \tag{3}$$

and the marginal probabilities

$$q_{i|\{i\}} = q_{i,\{i\}}, \quad q_{i|\{1,2\}} = q_{i,\{1,2\}} + q_{\{1,2\},\{1,2\}}. \tag{4}$$

Assume that at the end of each slot, the receiver gives an instantaneous feedback of all the packets that were successfully received to all the users. The users remove successful packets from their buffers while unsuccessful packets are retained. Let N_i^t denote the queue length of the ith user at the beginning of time slot t, the queue evolution function for the ith ($i \in \{1, 2, \ldots, N\}$) queue is given by [8, 10]

$$N_i^{t+1} = \left[N_i^t - Y_i^t\right]^+ + \beta_i^t, \tag{5}$$

where β_i^t is the number of arrivals during the tth slot to the ith user with $E(\beta_i^t) = \lambda_i < \infty$, Y_i^t is the Bernoulli random variable denoting the departure from queue i in time slot t, and $[x]^+ = \max(0, x)$.

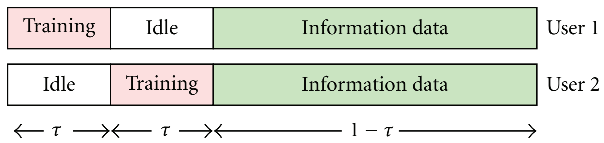

Figure 1: Illustration of the hybrid ALOHA slot structure for $M = 2$, in the case that two users ($N = 2$) transmit their training sequences at nonoverlapping pilot subslots.

2.2. Hybrid ALOHA. The proposed hybrid ALOHA protocol aims at improving MPR capability by allowing conditional collision-free channel estimation and simultaneous transmission. In hybrid ALOHA, each slot contains one *data sub-slot* and multiple *pilot sub-slots*, and each user can randomly select one of the pilot sub-slots to transmit his/her training sequence. In other words, idle pilot sub-slot(s) are introduced to make it possible for different users to transmit their training sequences at nonoverlapping sub-slots, whereby collision-free channel estimation could be achieved. If the physical layer can accommodate M users, (i.e., given reasonably accurate channel estimation, the user's packet can be successfully decoded if and only if there are no more than M simultaneous users then the hybrid ALOHA slot has M pilot sub-slots, which implies that $M - 1$ idle sections are inserted to each traditional ALOHA slot. (Signal processing plays a key role in separating/retrieving multiple users' signals. The principal (and not exclusive) example of systems with multiple access interference arises in networks using code-division multiple-access (CDMA) for uplink channel sharing. The area of study that deals with signal separation is *multiuser detection*. Various techniques, including linear multiuser detectors such as the zero-forcing and MMSE, as well as nonlinear interference cancellation techniques, have been proposed in the literature and implemented in real systems [16, 17].),

Figure 1 illustrates the slot structure of hybrid ALOHA in the case of $M = 2$. Throughout the paper, without loss of generality, we investigate this specific case for description convenience. Each slot has $M+1 = 3$ sub-slots. The preceding two sub-slots, each having a length of τ, are the "pilot sub-slots" reserved for training sequences. When a user is involved in a transmission, we assume that the selection of the pilot sub-slots is random and of equal probability. The *information data* is always transmitted in the last sub-slot referred to as the "data sub-slot." We assume that the length of the data sub-slot is $1 - \tau$ with $\tau \ll 1$. The length of the traditional ALOHA slot is used as the reference time unit, denoted as 1, which consists of a training sub-slot of length τ and a data sub-slot of length $1 - \tau$.

3. Sufficient Condition of Stability for $N > 2$

In this paper, the sufficient condition for stability of $N > 2$ hybrid ALOHA systems is derived based on the simplistic model which assumes that *the users who collide in pilot sub-slots fail in transmission whereas the users who have collision-free channel estimation survive, under the condition that there are no more than M users transmitting simultaneously in one slot*. More general results can be referred to [15].

The mathematical definition of *stability* is given as follows [8].

Definition 1. A multidimensional stochastic process $\mathbf{N}^t = (N_1^t, \ldots, N_N^t)$ is said to be *stable* if for $\mathbf{x} \in \mathbb{N}^N$ the following holds:

$$\lim_{t\to\infty} Pr\{\mathbf{N}^t < \mathbf{x}\} = F(\mathbf{x}), \qquad \lim_{\mathbf{x}\to\infty} F(\mathbf{x}) = 1, \tag{6}$$

where $F(\mathbf{x})$ is the limiting distribution function.

For an N-user system, the **stability region** is defined as the set of arrival rates $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_N]$ for which there exists a transmission probability vector $\mathbf{p} = [p_1, p_2, \ldots, p_N]$ such that the queues in the system are stable. When $N = 2$, suppose that the arrival rates for the two users are λ_1 and λ_2 (packets per slot), and their transmission probabilities are p_1 and p_2, respectively. We have derived the following result in [15].

Lemma 1. *For a fixed transmission probability vector $\mathbf{p} = [p_1, p_2]$, the stability region of hybrid ALOHA is given by*

$$\lambda_1 \le p_1 - \frac{p_1\lambda_2}{2 - p_1}, \quad \text{for } \lambda_2 \le p_2 - \frac{p_1 p_2}{2}, \tag{7}$$

$$\lambda_2 \le p_2 - \frac{p_2\lambda_1}{2 - p_2}, \quad \text{for } \lambda_1 \le p_1 - \frac{p_1 p_2}{2}, \tag{8}$$

where λ_1 and λ_2 are the arrival rates for the two users, respectively.

When $N > 2$, we will use the method of stochastic dominance [6] and the Loynes Theorem [18] to derive the sufficient condition for stability of hybrid ALOHA. (A real random variable X is said to stochastically dominate a real random variable Y if for all $z \in \mathbb{R}$, $Pr\{X > z\} \ge Pr\{Y > z\}$. This dominance is denoted by $X \ge_{st} Y$.) To achieve this goal, we construct a modified system as follows. Let $\mathcal{P} = \{\mathcal{S}, \mathcal{U}\}$ be a partition of $\mathcal{N}$ such that users in $\mathcal{S} \neq \mathcal{N}$ work exactly in the same manner as in the original system, while users in $\mathcal{U}$*persistently* transmit *dummy* packets even if their queues are empty. Users in $\mathcal{U}$ are called *persistent* and those in $\mathcal{S}$*nonpersistent*. Such modified system is denoted by $\Theta^{\mathcal{P}}$. Let $\mathbf{N}_{\mathcal{P}}^t = (\mathbf{N}_{\mathcal{S}}^t, \mathbf{N}_{\mathcal{U}}^t)$ denote the queue lengths in $\Theta^{\mathcal{P}}$ and it can be proved that $\mathbf{N}_{\mathcal{P}}^t$ stochastically dominates $\mathbf{N}^t$ of the original system provided that the initial conditions are identical [4, 5].

By the construction above, the process $\mathbf{N}_{\mathcal{S}}^t$ is an $|\mathcal{S}|$-dimensional Markov chain that mimics the behavior of the original system. Note that the system consisting of users in $\mathcal{S}$ forms a smaller copy of the original system with modified reception probabilities. Generally, for any $\mathcal{S}' \subseteq \mathcal{S}$ and $i \in \mathcal{S}'$, the modified reception probabilities for the smaller system consisting of the stand-alone nonpersistent set $\mathcal{S}$ are given by [10]

$$q_{i|\mathcal{S}'}^{\mathcal{S}} = \sum_{\mathcal{T}\subseteq\mathcal{U}} \left[\prod_{j\in\mathcal{T}} p_j \prod_{k\in\mathcal{U}\setminus\mathcal{T}} (1 - p_k)\right] q_{i|\mathcal{S}'\cup\mathcal{T}}^{\mathcal{N}}. \tag{9}$$

Induction arguments can then be applied to establish the stability condition. We further assume that the Markov chain $\mathbf{N}_{\mathcal{S}}^t$ is stationary and ergodic. We denote the stationary version as $\overline{\mathbf{N}}_{\mathcal{S}}^0$ to indicate that the process starts from the stationary distribution.

Let $Y_i^t(\mathcal{P})$ be the departure process from the ith queue in the dominant system $\Theta^{\mathcal{P}}$, then we have

$$Y_i^t(\mathcal{P}) = \sum_{j=1}^{2}\left[1-\chi\left(R_{i,\bar{j}}^t\right)\right]R_{i,j}^t \cdot \mathbf{1}\left[\sum_{k\in\mathcal{S}\setminus\{i\}} R_{k,j}^t\chi\left(N_k^t\right) + \sum_{k\in\mathcal{U}\setminus\{i\}} R_{k,j}^t = 0\right] \cdot \mathbf{1}\left[\sum_{k\in\mathcal{S}\setminus\{i\}} R_{k,\bar{j}}^t\chi\left(N_k^t\right) + \sum_{k\in\mathcal{U}\setminus\{i\}} R_{k,\bar{j}}^t \le 1\right], \tag{10}$$

where j is the index of the pilot sub-slot and $\bar{j} = \{1,2\}\setminus\{j\}$; $\mathbf{1}[\cdot]$ is the indicator function; $R_{i,j}^t$ is the i.i.d Bernoulli $(p_{i,j})$ random variable for $1 \le i \le N$, indicating the outcomes of transmission attempts, where $p_{i,j}$ is the transmission probability of the training sequence at sub-slot j for user i, hence the transmission probability of user i in one slot is given by $p_i = \sum_{j=1}^{2} p_{i,j}$. For hybrid ALOHA, since one user cannot transmit in both pilot sub-slots, $R_{i,\bar{j}}^t = 1 - R_{i,j}^t$. The function $\chi(k)$ is defined as

$$\chi(k) = \begin{cases} 0, & k = 0, \\ 1, & k > 0. \end{cases} \tag{11}$$

The two indicator functions represent, respectively, that when there is a transmission occurring in sub-slot i, no others can transmit in that same sub-slot, and at most one other user can transmit in the other sub-slot.

Given that $Y_i^t(\mathcal{P})$ is stationary, we denote $P_{\mathrm{succ}}^i(\mathcal{P}) = E[Y_i^t(\mathcal{P})]$ as the probability of a successful transmission from the ith user in system $\Theta^{\mathcal{P}}$, which is given by

$$\begin{aligned} P_{\mathrm{succ}}^i(\mathcal{P}) = {}& p_i \prod_{k\in\mathcal{U}\setminus\{i\}}(1-p_k) \\ &\times \sum_{\mathbf{z}_{\mathcal{S}}\in\{0,1\}^{|\mathcal{S}|}}\left(Pr\left\{\chi\left(\overline{\mathbf{N}}_{\mathcal{S}}^0\right) = \mathbf{z}_{\mathcal{S}}\right\}\prod_{k\in\mathcal{S}\setminus\{i\}}(1-p_k)^{z_k}\right) \\ &+ \left(\sum_{j=1}^{2} p_{i,j}\sum_{k\in\mathcal{U}\setminus\{i\}}\left[p_{k,\bar{j}}\prod_{k'\in\mathcal{U}\setminus\{i,k\}}(1-p_{k'})\right]\right) \\ &\cdot \sum_{\mathbf{z}_{\mathcal{S}}\in\{0,1\}^{|\mathcal{S}|}}\left(Pr\left\{\chi\left(\overline{\mathbf{N}}_{\mathcal{S}}^0\right) = \mathbf{z}_{\mathcal{S}}\right\}\prod_{k\in\mathcal{S}\setminus\{i\}}(1-p_k)^{z_k}\right) \\ &+ \prod_{k\in\mathcal{U}\setminus\{i\}}(1-p_k) \\ &\times \left(\sum_{j=1}^{2} p_{i,j}\sum_{\mathbf{z}_{\mathcal{S}}\in\{0,1\}^{|\mathcal{S}|}} Pr\left\{\chi\left(\overline{\mathbf{N}}_{\mathcal{S}}^0\right) = \mathbf{z}_{\mathcal{S}}\right\} \cdot \left[\sum_{k\in\mathcal{S}\setminus\{i\}} p_{k,\bar{j}}^{z_k}\prod_{k'\in\mathcal{S}\setminus\{i,k\}}(1-p_{k'})^{z_{k'}}\right]\right), \end{aligned} \tag{12}$$

where $\chi(\overline{\mathbf{N}}_{\mathcal{S}}^0) = (\chi(\overline{N}_{i_1}^0),\ldots,\chi(\overline{N}_{i_{|\mathcal{S}|}}^0))$ with $i_k \in \mathcal{S}$ for all $k = 1,\ldots,|\mathcal{S}|$. The first term of the right-hand side of (12) represents the probability of successful transmission of user i when no one else transmits. The second and third terms represent the probabilities of successful transmission of user i when there is another user involved in the transmission. The second term corresponds to the case when the extra user belongs to $\mathcal{U}$, and the third term corresponds to the case when the extra user belongs to $\mathcal{S}$.

Let $\mathfrak{S}_{H-\mathrm{ALOHA}}$ be the stability region of the original hybrid ALOHA system. Let $\mathfrak{S}_{\mathcal{N}}$ and $\mathfrak{S}_{\mathcal{S}}$ be, respectively, stability regions for dominant system $\Theta^{\mathcal{P}}$ and the system consisting of nonpersistent users $\mathcal{S}$ under the partition of $\mathcal{P} = (\mathcal{S},\mathcal{U})$. Defining a region

$$\mathfrak{S}_{\mathcal{N}} = \bigcup_{\mathcal{P}}\Big\{\boldsymbol{\lambda}_{\mathcal{N}} = \{\lambda_1,\ldots,\lambda_N\} : \lambda_k < P_{\mathrm{succ}}^k(\mathcal{P})\ \forall\ k\in\mathcal{U}, \quad \boldsymbol{\lambda}_{\mathcal{S}} \in \mathfrak{S}_{\mathcal{S}}\Big\}, \tag{13}$$

we have the following result.

Proposition 1 (see [15]). *If $\boldsymbol{\lambda}_{\mathcal{N}} = \{\lambda_1,\ldots,\lambda_N\} \in \mathfrak{S}_{\mathcal{N}}$, then hybrid ALOHA is stable, that is, $\mathfrak{S}_{\mathcal{N}} \subseteq \mathfrak{S}_{H-ALOHA}$.*

Evaluation of the stability condition for the general N is nontrivial, if not impossible, due to the difficulty in evaluating the stationary distribution of the queues with arbitrary input distributions. In the next section, the specific case of $N = 3$ is inspected and the stability inner bounds of such a case are derived.

4. Special Case: Stability Inner Bounds for $N = 3$

4.1. Inner Bounds Characterization. Recall that $\mathcal{P} = \{\mathcal{S},\mathcal{U}\}$ is a partition of $\mathcal{N}$. Consider the three partitions $\mathcal{P}_i = \{\mathcal{N}_i,\{i\}\}$, where $\mathcal{N}_i = \mathcal{N}\setminus\{i\}$, $i = 1,2,3$. Define $\mathcal{P}_{\bar{i}} = \{\{i\},\mathcal{N}_i\}$. From (12), it can be verified that $P_{\mathrm{succ}}^i(\mathcal{P}_{\bar{i}}) \le P_{\mathrm{succ}}^i(\mathcal{P}_i)$ (Appendix A). This result is intuitively correct as well because chances of collision for user i in system $\mathcal{P}_i$ are smaller than that in system $\mathcal{P}_{\bar{i}}$, due to the way of system construction. Hence, it follows from Proposition 1 that the stability inner bound $\mathcal{R}$ is the union of three regions $\mathcal{R}_i$ corresponding to $\mathcal{P}_i$, $i = 1,2,3$, respectively. In what follows, we closely inspect region $\mathcal{R}_3$ that is corresponding to $\mathcal{P}_3 = \{\{1,2\},\{3\}\}$. The other two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ can be easily obtained through similar procedures.

Let N_i^t and β_i^t denote, respectively, the queue length and the number of arriving packets of user $i (i = 1,2)$ in time slot t. Let $F_3(x,y)$ be the moment generating function of the joint arrival process for users 1 and 2. Thus, for $|x| \le 1$, $|y| \le 1$, $t \in \mathbb{N}$,

$$F_3(x,y) = E\left(x^{\beta_1^t}y^{\beta_2^t}\right) = \left(x\lambda_1 + \overline{\lambda_1}\right)\left(y\lambda_2 + \overline{\lambda_2}\right), \tag{14}$$

where $\overline{\lambda_i} = 1 - \lambda_i$, for $i = 1,2$.

The investigated system model implies that (N_1^t, N_2^t) is an irreducible, aperiodic Markov chain. Hence the stability of the system is equivalent to existence of a unique stationary (limiting) distribution. Let $G_3(x, y)$ be the moment generating function of the joint stationary queue process, namely,

$$G_3(x, y) = \lim_{t \to \infty} E\left(x^{N_1^t} y^{N_2^t}\right), \tag{15}$$

which is analytic with respect to x and y whenever $|x|, |y| \leq 1$.

In Appendix B, we derive the following functional equation:

$$\begin{aligned} K(x, y) G_3(x, y) = {} & a(x, y) G_3(0, y) + b(x, y) G_3(x, 0) \\ & + c(x, y) G_3(0, 0), \end{aligned} \tag{16}$$

with

$$\begin{aligned} K(x, y) &= \frac{1}{\left(x\lambda_1 + \overline{\lambda_1}\right)\left(y\lambda_2 + \overline{\lambda_2}\right)} - \frac{p_1 \overline{p}_2 (1 - p_3/2)}{x} \\ &\quad - \frac{p_2 \overline{p}_1 (1 - p_3/2)}{y} - \frac{p_1 p_2 \overline{p}_3}{2xy} - \Delta, \\ a(x, y) &= \frac{p_1 p_2 (1 - p_3/2)}{y} + \frac{p_1 \overline{p}_2 (1 - p_3/2)}{x} - \frac{p_1 p_2 \overline{p}_3}{2xy} \\ &\quad + p_1 \left(1 - \frac{p_3}{2}\right)(1 - 2p_2) + \frac{p_1 p_2 \overline{p}_3}{2}, \\ b(x, y) &= \frac{p_1 p_2 (1 - p_3/2)}{x} + \frac{p_2 \overline{p}_1 (1 - p_3/2)}{y} - \frac{p_1 p_2 \overline{p}_3}{2xy} \\ &\quad + p_2 \left(1 - \frac{p_3}{2}\right)(1 - 2p_1) + \frac{p_1 p_2 \overline{p}_3}{2}, \\ c(x, y) &= \frac{p_1 p_2 \overline{p}_3}{2xy} - \frac{p_1 p_2 (1 - p_3/2)}{y} - \frac{p_1 p_2 (1 - p_3/2)}{x} \\ &\quad + \frac{1}{2} p_1 p_2 (3 - p_3), \end{aligned} \tag{17}$$

where $\Delta = 1 - (1 - p_3/2)(p_1 \overline{p}_2 + \overline{p}_1 p_2) - p_1 p_2 \overline{p}_3/2$.

Define $P_3(z_1, z_2) \doteq Pr\{\chi(\overline{N}_1) = z_1, \chi(\overline{N}_2) = z_2\}$ with user 3 being the persistent one. Then from (B.5)– (B.8) we have

$$\begin{aligned} P_3(0, 0) &= G_3(0, 0), \\ P_3(1, 0) &= G_3(1, 0) - G_3(0, 0), \\ P_3(0, 1) &= G_3(0, 1) - G_3(0, 0), \\ P_3(1, 1) &= 1 + G_3(0, 0) - G_3(0, 1) - G_3(1, 0). \end{aligned} \tag{18}$$

Using these relations and (12) and assuming $p_{i,j} = p_i/2$ for $i = 1, 2, 3$, $j = 1, 2$, we can obtain the following results:

$$\begin{aligned} P_{\text{succ}}^1(\mathcal{P}_3) &= p_1 \overline{p}_3 [1 - (1 - G_3(1, 0) p_2)] \\ &\quad + (p_{1,1} p_{3,2} + p_{1,2} p_{3,1})[1 - (1 - G_3(1, 0)) p_2] \\ &\quad + \overline{p}_3 [(p_{1,1} p_{2,2} + p_{1,2} p_{2,1})(P_3(0, 1) + P_3(1, 0))] \\ &= p_1 \left[1 - \frac{1}{2} p_3 - \frac{1}{2} p_2 (1 - G_3(1, 0))\right], \\ P_{\text{succ}}^2(\mathcal{P}_3) &= p_2 \overline{p}_3 [1 - (1 - G_3(0, 1) p_1)] \\ &\quad + (p_{2,2} p_{3,1} + p_{2,1} p_{3,2})[1 - (1 - G_3(0, 1)) p_1] \\ &\quad + \overline{p}_3 [(p_{2,1} p_{1,2} + p_{2,2} p_{1,1})(P_3(0, 1) + P_3(1, 0))] \\ &= p_2 \left[1 - \frac{1}{2} p_3 - \frac{1}{2} p_1 (1 - G_3(0, 1))\right], \\ P_{\text{succ}}^3(\mathcal{P}_3) &= p_3 \left[P_3(0, 0) + \left(1 - \frac{1}{2} p_1\right) P_3(1, 0) \right. \\ &\quad + \left(1 - \frac{1}{2} p_2\right) P_3(0, 1) \\ &\quad \left. + \left(1 - \frac{1}{2} p_1 - \frac{1}{2} p_2\right) P_3(1, 1)\right] \\ &= p_3 \left[1 - \frac{1}{2} p_1 (1 - G_3(0, 1)) - \frac{1}{2} p_2 (1 - G_3(1, 0))\right]. \end{aligned} \tag{19}$$

Stability region $\mathcal{R}_3$ can then be characterized by $\mathcal{R}_3 = \{\lambda_i < P_{\text{succ}}^i(\mathcal{P}_3), i = 1, 2, 3\}$. The other two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ can be similarly calculated and the following result is then readily in form.

Proposition 2. *For $N = 3$, hybrid ALOHA is stable in the region $\mathcal{R} = \bigcup_{k=1}^3 \mathcal{R}_k$, with*

$$\mathcal{R}_k = \left\{\lambda_i < P_{\text{succ}}^i(\mathcal{P}_k), i = 1, 2, 3\right\}, \tag{20}$$

where $P_{\text{succ}}^i(\mathcal{P}_k)$ for $k = 3, i = 1, 2, 3$, are shown in (19), and it can be calculated in the same manner for $k = 1, 2, i = 1, 2, 3$.

It can be seen that the probability of success $P_{\text{succ}}^i(\mathcal{P}_k)$ depends explicitly on $G_k(1, 0)$ and $G_k(0, 1)$. These functions are generally nonlinear functions of the input rates. It will be shown in the sequel that finding out the explicit expression of these functions reduces to solving a general Riemann-Hilbert problem. The analysis follows the procedures in [1, 19].

4.2. Analysis of the Kernel $K(x, y)$. The analysis of the kernel $K(x, y)$ is the key to solving the functional equation (16). For description simplicity, in what follows we omit the subscript of the function $G_k(x, y)$. Since $G(x, y)$ is analytic in $|x| < 1, |y| < 1$ and continuous in $|x| \leq 1, |y| \leq 1$, the right-hand side of (16) must vanish whenever the "kernel" $K(x, y)$ vanishes for $|x| \leq 1, |y| \leq 1$.

Rewrite the kernel $K(x, y)$ given in (17) as

$$K(x,y) = \frac{1}{\left(x\lambda_1 + \overline{\lambda_1}\right)\left(y\lambda_2 + \overline{\lambda_2}\right)} - \frac{p_1\overline{p}_2(1-p_3/2)}{x} - \frac{p_2\overline{p}_1(1-p_3/2)}{y} - \frac{p_1p_2\overline{p}_3}{2xy} - \Delta. \tag{21}$$

Solving for x the equation $K(x, y) = 0$, we will have a root $x = f_x(y)$ satisfying

$$A(y)x^2 + B(y)x + C(y) = 0, \tag{22}$$

where, by defining $\Delta = 1-(1-p_3/2)(p_1\overline{p}_2+\overline{p}_1p_2)-p_1p_2\overline{p}_3/2$, we have

$$\begin{aligned}
A(y) &= \lambda_1\left[\overline{p}_1p_2\left(1-\frac{p_3}{2}\right)\left(\lambda_2+\frac{\overline{\lambda}_2}{y}\right)+\Delta\left(y\lambda_2+\overline{\lambda}_2\right)\right],\\
B(y) &= \left[\lambda_1p_1\overline{p}_2\left(1-\frac{p_3}{2}\right)+\overline{\lambda}_1\Delta\right]\left(y\lambda_2+\overline{\lambda}_2\right)\\
&\quad+\left[\overline{\lambda}_1\overline{p}_1p_2\left(1-\frac{p_3}{2}\right)+\lambda_1\frac{p_1p_2\overline{p}_3}{2}\right]\left(\lambda_2+\frac{\overline{\lambda}_2}{y}\right)-1,\\
C(y) &= \overline{\lambda}_1p_1\overline{p}_2\left(1-\frac{p_3}{2}\right)\left(y\lambda_2+\overline{\lambda}_2\right)+\overline{\lambda}_1\frac{p_1p_2\overline{p}_3}{2}\left(\lambda_2+\frac{\overline{\lambda}_2}{y}\right).
\end{aligned} \tag{23}$$

Therefore,

$$f_x(y) = \frac{-B(y) \pm \sqrt{D(y)}}{2A(y)}, \tag{24}$$

with $D(y) = B(y)^2 - 4A(y)C(y)$. Defining $t(y, \phi) = -B(y) - 2\cos(\phi)\sqrt{A(y)C(y)}$, we have the following lemma.

Lemma 2. *For $\phi \in [0, 2\pi]$, the equation $t(y, \phi) = 0$ has exactly two real roots $y = r_1(\phi)$ and $y = r_2(\phi)$ which satisfy $0 < r_1(\phi) < 1 < r_2(\phi)$.*

Proof. Please refer to Appendix C. □

Since $D(y) = t(y, 0)t(y, \pi)$, it is readily seen that $y_1 = r_1(\pi), y_2 = r_1(0), y_3 = r_2(0), y_4 = r_2(\pi)$ are the four zeros of $D(y)$ (branch points of $x(y)$) satisfying $0 < y_1 < y_2 < 1 < y_3 < y_4$. This result is evidently also valid for the branch points x_1, x_2, x_3, x_4 of the function $y(x)$. The following lemma then holds.

Lemma 3. *The equation $K(x, y) = 0$ has one root $x = k(y)$ which is an analytic algebraic function of y in the whole complex plane cut along $[y_1, y_2] \cup [y_3, y_4]$. Moreover, $|k(y)| \leq 1$ for $|y| = 1$.*

Similar propositions apply to $y(x)$. That is, there exists $y = h(x)$ such that $K(x, h(x)) = 0$ with $|h(x)| \leq 1$ for $|x| = 1$.

Proof. Please refer to Appendix D. □

Defining

$$\rho(\phi) \doteq \left(\frac{\overline{\lambda}_1\left[p_1\overline{p}_2(1-p_3/2)r_1(\phi)+p_1p_2\overline{p}_3/2\right)\right]}{\lambda_1\left[\Delta r_1(\phi)+\overline{p}_1p_2(1-p_3/2)\right]}\right)^{1/2}, \tag{25}$$

we then have the following result.

Lemma 4. *One has $k(r_1(\phi)) = \rho(\phi)e^{i\phi}$ for $\phi \in [0, 2\pi]$.*

Proof. From Lemma 3, $f_x(y) = k(y)$ is the algebraic branch of $K(x, y) = 0$ such that $|x(y)| \leq 1$ for $|y| = 1$. Denote $k_c(y)$ as the other root of equation $K(x, y) = 0$. It is shown that for $y \in \mathbb{C} \setminus [y_1, y_2] \cup [y_3, y_4]$, the minus and plus signs in (24) correspond to $k(y)$ and $k_c(y)$, respectively (compute $k(1)$ and $k_c(1)$). Observe that $y = r_1(\phi)$ sweeps twice the cut $[y_1, y_2]$ as ϕ traverses the interval $[0, 2\pi]$, then $k(r_1(\phi))$ and $k_c(r_1(\phi))$ are two conjugate complex numbers satisfying

$$k(r_1(\phi))k_c(r_1(\phi)) = \frac{C(r_1(\phi))}{A(r_1(\phi))} = |\rho(\phi)|^2. \tag{26}$$

From the definition of the algebraic branch, it is shown that $k(r_1(\phi)) = \rho(\phi)e^{i\phi}, \phi \in [0, 2\pi]$. □

The image of the cut $[y_1, y_2]$ under the mapping $x = k(y)$ is then denoted as $L_x \doteq \{x \in \mathbb{C} : x = \rho(\phi)e^{i\phi}, \phi \in [0, 2\pi]\}$, which is a smooth closed contour enclosing 0.

4.3. Reduction to the Riemann-Hilbert Problem. It will be shown in this subsection that the problem of finding the expressions of function $G(x, 0)$ and $G(0, y)$ reduces to solving a general Riemann-Hilbert boundary value problem.

Riemann-Hilbert Problem. Let L^+ be a finite or infinite region, bounded by a smooth contour L. It is required to find a function $\Phi(z)$, holomorphic in L^+ and continuous in $L^+ \cup L$, satisfying the boundary condition

$$\Re[U(z)\Phi(z)] = V(z) \text{ on L}, \tag{27}$$

where $U(z), V(z)$ are continuous functions given on L.

The formulation of the boundary value problem is illustrated below.

For pairs (x, y) satisfying $K(x, y) = 0$ and $|x| \leq 1$, $|y| \leq 1$, we should have

$$a(x, y)G(0, y) + b(x, y)G(x, 0) + c(x, y)G(0, 0) = 0. \tag{28}$$

Defining $D = \{y \in \mathbb{C} : |y| \leq 1, |k(y)| \leq 1\}$ and $D_c = \{y \in \mathbb{C} : |y| \leq 1, |k(y)| > 1\}$, we have, for $y \in D$,

$$\begin{aligned}
&a(k(y), y)G(0, y) + b(k(y), y)G(k(y), 0)\\
&\quad= -c(k(y), y)G(0, 0).
\end{aligned} \tag{29}$$

$G(0, y)$ and $G(x, 0)$ are analytic in $D \setminus [y_1, y_2]$. When $|y| \leq 1$, $k(y)$ is in the region containing the curve L_x. Then (29) can be used to continue $G(x, 0)$ as a meromorphic function up to

L_x. The eventual poles of $G(x,0)$ are the zeros of $b(k(y),y)$ for $y \in D_c$.

The definition entails that the power series expansion of $G(0,y)$ has positive coefficients for $|y| \le 1$. Hence, we have $\Re\{iG(0,y)\} = 0$ for $y \in [y_1, y_2]$. It then follows that

$$\Re\left\{\frac{ib(x,h(x))}{a(x,h(x))}G(x,0)\right\} = \Re\left\{\frac{-ic(x,h(x))}{a(x,h(x))}G(0,0)\right\}, \quad \text{for } x \in L_x. \tag{30}$$

If $a(x,h(x)) = 0$ has a set $\mathcal{A}$ of solutions on L_x, and a_i denotes the ith solution of multiplicity m_i, we can reformulate (30) into

$$\begin{aligned}\Re&\left\{\frac{ib(x,h(x))\prod_{a_i\in\mathcal{A}}(x-a_i)^{m_i}}{a(x,h(x))}G(x,0)\right\}\\ &= \Re\left\{\frac{-ic(x,h(x))\prod_{a_i\in\mathcal{A}}(x-a_i)^{m_i}}{a(x,h(x))}G(0,0)\right\},\end{aligned} \tag{31}$$

where $m_i = 0$ means root a_i does not exist.

Let

$$U(x) = \frac{b(x,h(x))\prod_{a_i\in\mathcal{A}}(x-a_i)^{m_i}}{a(x,h(x))}, \tag{32}$$

$$V(x) = \Re\left\{\frac{-ic(x,h(x))\prod_{a_i\in\mathcal{A}}(x-a_i)^{m_i}}{a(x,h(x))}G(0,0)\right\}. \tag{33}$$

and denote

$$\Phi(x) = G(x,0). \tag{34}$$

The problem of finding the expression of $G(x,0)$ then reduces to finding a function $\Phi(x)$ which is analytic in L_x^+, continuous in $L_x \cup L_x^+$, and satisfies

$$\Re\{iU(x)\Phi(x)\} = V(x), \quad \text{for } x \in L_x. \tag{35}$$

This is a typical Riemann-Hilbert boundary value problem, and it will be demonstrated in the next subsection how to obtain the solution.

Remark. In the case where the unit disk is not entirely contained in L_x^+, we need to analytically continue the solution $\Phi(x)$ to the unit circle to obtain $G(x,0)$ for all $|x| \le 1$. The expression of function $G(0,y)$ can be obtained in the same manner.

4.4. Solution to the Riemann-Hilbert Problem (35). The solution of (35) can be directly obtained as in [20, pages 99–107], whenever the contour L_x is a unit circle. The problem is more complicated when L_x is arbitrary. Fortunately, Riemann's mapping theorem guarantees the existence of a conformal mapping which maps L_x conformally onto the unit circle. Such a mapping can be fulfilled at the aid of the Theodorsen's procedure [21, pages 70–73], which performs the inverse mapping from the unit circle to L_x and is stated in the following lemma [21].

Lemma 5. *The conformal mapping f_0 of the unit circle $z = e^{it}, t \in [0,2\pi]$ onto the curve $L_x = \{x : x = \rho(\phi)e^{i\phi}, \phi \in [0,2\pi]\}$ is defined as*

$$f_0\left(e^{it}\right) = \rho(\phi(t))e^{i\phi(t)}, \tag{36}$$

where f_0 is assumed to be normalized by $f_0(0) = 0$ and $f_0'(0) > 0$. For $|z| < 1$, $f_0(z)$ is uniquely determined by

$$f_0(z) = z\exp\left[\frac{1}{2\pi}\int_0^{2\pi}\log\rho(\phi(t))\frac{e^{it}+z}{e^{it}-z}dt\right], \quad \text{for } |z| < 1, \tag{37}$$

where $\phi(t)$ satisfies

$$\phi(t) = t - \frac{1}{2\pi}\int_0^{2\pi}\log\rho(\phi(\omega))\cot\frac{1}{2}(\omega - t)d\omega, \quad 0 \le t \le 2\pi. \tag{38}$$

This is Theodorsen's integral equation for $\phi(t)$; it is a nonlinear, singular integral equation.

The details of solving the equations in Lemma 5 can be found in [21] and will not be discussed in this paper. We denote $f(z)$ as the inverse of $f_0(z)$. Using Lemma 5 and the methods in [21, pages 68–69], and [20, pages 99–107], we can find the solution to the Riemann-Hilbert problem (35) as below.

Define the index of the nonhomogeneous Riemann-Hilbert boundary value problem (35) as

$$\kappa = \frac{-1}{\pi}\arg[U(x)]_{x\in L_x}, \tag{39}$$

where $\arg[U(x)]_{x\in L_x}$ is the variation of the argument of the function $U(x)$ when x moves along the contour L_x in the positive direction.

We present the solution to (35) for $\kappa \ge 0$ in what follows.

The solution is given by $\Phi(x) = H(f(x))$, where

$$\begin{aligned}H(z) = \frac{z^\kappa S(z)}{2\pi}&\int_{|t|=1}\frac{t^{-\kappa}V(t)dt}{U(t)S^+(t)t}\\ &- \left\{\int_{|t|=1}\frac{V(t)dt}{U(t)S^+(t)(t-z)}\right.\\ &\left.+ z^\kappa\int_{|t|=1}\frac{t^{-\kappa}V(t)dt}{U(t)S^+(t)(t-z)}\right\}\frac{S(z)}{2\pi}.\end{aligned} \tag{40}$$

In the above equation,

$$S(z) = C\cdot e^{\Gamma(z)}, \tag{41}$$

where C is a nonzero constant and

$$\Gamma(z) = \frac{1}{2\pi i}\int_{|t|=1}\frac{\log[t^{-\kappa}J(t)]dt}{t-z}, \tag{42}$$

with

$$J(t) = \frac{\overline{iU(t)}}{iU(t)},$$

$$U(t) = \frac{b(f_0(t), h(f_0(t)))\prod_{a_i \in \mathcal{A}}(f_0(t) - a_i)^{m_i}}{a(f_0(t), h(f_0(t)))},$$

$$V(t) = \Re\left\{\frac{-ic(f_0(t), h(f_0(t)))\prod_{a_i \in \mathcal{A}}(f_0(t) - a_i)^{m_i}}{a(f_0(t), h(f_0(t)))} \times G(0,0)\right\}. \tag{43}$$

For $t_0 \in L$, $S^+(t_0)$ is defined to be the limit when t approaches t_0 along any path, which remains, however, on the left of L, that is,

$$S^+(t_0) = \lim_{t \to t_0, t \in L^+} S(t). \tag{44}$$

Applying the Plemelj-Sokhotski formulas [21, page 32], it can be shown that, in (40),

$$S^+(t_0) = \frac{1}{2}S(t_0) + \frac{1}{2\pi i}\int_{|t|=1}\frac{S(t)}{t - t_0}dt. \tag{45}$$

Finally, $G(x, 0)$ is obtained through (34).

On the other hand, $G(0, y)$ can be computed through the similar procedures, and consequently the stability bound in Proposition 2 can be explicitly determined.

5. Conclusions

In this paper, we studied the stability region of the hybrid ALOHA protocol for the $N > 2$ case. The method of stochastic dominance was applied to obtain the results. Specifically, by constructing modified systems dominating the original system, and by means of mathematical induction, the sufficient condition for stability of the $N > 2$ user system was obtained. Furthermore, we elaborated on the characterization of stability inner bounds for the $N = 3$ case, for which the results were obtained by solving a nonhomogeneous Riemann-Hilbert boundary value problem.

Appendices

A. Proof of $P^i_{\text{succ}}(\mathcal{P}_{\bar{i}}) \leq P^i_{\text{succ}}(\mathcal{P}_i)$ in Section 4.1

According to the partitions that $\mathcal{P}_i = \{\mathcal{N}_i, \{i\}\}$ and $\mathcal{P}_{\bar{i}} = \{\{i\}, \mathcal{N}_i\}$, we can see that the second term of (12) vanishes for $P^i_{\text{succ}}(\mathcal{P}_i)$ and the third term vanishes for $P^i_{\text{succ}}(\mathcal{P}_{\bar{i}})$. The following inequality then holds:

$$\begin{aligned}
P^i_{\text{succ}}(\mathcal{P}_i) &= p_i \sum_{\mathbf{z}_{\mathcal{S}} \in \{0,1\}^{|\mathcal{S}|}}\left(Pr\left\{\chi\left(\overline{\mathbf{N}}^0_{\mathcal{S}}\right) = \mathbf{z}_{\mathcal{S}}\right\}\prod_{k \in \mathcal{N}_i}(1 - p_k)^{z_k}\right) \\
&\quad + \sum_{j=1}^{2} p_{i,j}\sum_{\mathbf{z}_{\mathcal{S}} \in \{0,1\}^{|\mathcal{S}|}} Pr\left\{\chi\left(\overline{\mathbf{N}}^0_{\mathcal{S}}\right) = \mathbf{z}_{\mathcal{S}}\right\} \\
&\quad \cdot \left[\sum_{k \in \mathcal{N}_i} p^{z_k}_{k,\bar{j}}\prod_{k' \neq k}(1 - p_{k'})^{z_{k'}}\right] \\
&\geq p_i \sum_{\mathbf{z}_{\mathcal{S}} \in \{0,1\}^{|\mathcal{S}|}}\left(Pr\left\{\chi\left(\overline{\mathbf{N}}^0_{\mathcal{S}}\right) = \mathrm{z}_{\mathcal{S}}\right\}\prod_{k \in \mathcal{N}_i}(1 - p_k)\right) \\
&\quad + \sum_{j=1}^{2} p_{i,j}\sum_{\mathbf{z}_{\mathcal{S}} \in \{0,1\}^{|\mathcal{S}|}} Pr\left\{\chi\left(\overline{\mathbf{N}}^0_{\mathcal{S}}\right) = \mathbf{z}_{\mathcal{S}}\right\} \\
&\quad \cdot \left[\sum_{k \in \mathcal{N}_i} p_{k,\bar{j}}\prod_{k' \neq k}(1 - p_{k'})\right] \\
&= p_i \prod_{k \in \mathcal{N}_i}(1 - p_k) + \sum_{j=1}^{2} p_{i,j}\sum_{k \in \mathcal{N}_i} p_{k,\bar{j}}\prod_{k' \neq k}(1 - p_{k'}) \\
&= P^i_{\text{succ}}(\mathcal{P}_{\bar{i}}).
\end{aligned} \tag{A.1}$$

This concludes the proof.

B. Formulation of Functional Equation (16)

Let $D_{10}(t)$ be a binary-valued random variable that takes value 1 if $N^t_1 > 0, N^t_2 = 0$, and the departure from queue 1 is successful. Similarly, $D_{01}(t)$ is a binary-valued random variable that takes value 1 if $N^t_2 > 0, N^t_1 = 0$, and the departure from queue 2 is successful. In the situation when both queues are nonempty, the binary-valued variables $D^i_{11}(t)$ for $i = 1, 2$ take value 1 when departure from queue i is successful. The recursive equations for N^t_i are given as

$$N^{t+1}_1 = \begin{cases} \beta^t_1, & N^t_1 = 0,\ N^t_2 = 0, \\ \beta^t_1 + N^t_1 - D_{10}(t), & N^t_1 > 0,\ N^t_2 = 0, \\ \beta^t_1, & N^t_1 = 0,\ N^t_2 > 0, \\ \beta^t_1 + N^t_1 - D^1_{11}(t), & N^t_1 > 0,\ N^t_2 > 0, \end{cases} \tag{B.1}$$

$$N^{t+1}_2 = \begin{cases} \beta^t_2, & N^t_1 = 0,\ N^t_2 = 0, \\ \beta^t_2, & N^t_1 > 0,\ N^t_2 = 0, \\ \beta^t_2 + N^t_2 - D_{01}(t), & N^t_1 = 0,\ N^t_2 > 0, \\ \beta^t_2 + N^t_2 - D^2_{11}(t), & N^t_1 > 0,\ N^t_2 > 0. \end{cases} \tag{B.2}$$

If a persistent user 3 exists, that is we investigate the partition $\mathcal{P}_3$, then from (B.1) and (B.2), we have

$$\begin{aligned}
&E\left(x^{N_1^{t+1}} y^{N_2^{t+1}}\right)\\
&= E\left(x^{\beta_1^t} y^{\beta_2^t}\right)\\
&\quad\times\Bigg\{E(\mathbf{1}[N_1^t=0,\ N_2^t=0])\\
&\quad+E\left(x^{N_1^t}\mathbf{1}[N_1^t>0,\ N_2^t=0]\right)\cdot\left[\frac{\Delta_1}{x}+1-\Delta_1\right]\\
&\quad+E\left(y^{N_2^t}\mathbf{1}[N_1^t=0,\ N_2^t>0]\right)\cdot\left[\frac{\Delta_2}{y}+1-\Delta_2\right]\\
&\quad+E\left(x^{N_1^t}y^{N_2^t}\mathbf{1}[N_1^t>0,\ N_2^t>0]\right)\\
&\quad\cdot\left[\frac{\Delta_3}{x}+\frac{\Delta_4}{y}+\frac{\Delta_5}{xy}+1-\Delta_3-\Delta_4-\Delta_5\right]\Bigg\},
\end{aligned}\tag{B.3}$$

where $\mathbf{1}[\cdot]$ is the indicator function and

$$\begin{aligned}
\Delta_1 &= p_1\left(\overline{p}_3 q_{\{1\},\ \{1\}}+p_3 q_{\{1\}|\{1,3\}}\right),\\
\Delta_2 &= p_2\left(\overline{p}_3 q_{\{2\},\{2\}}+p_3 q_{\{2\}|\{2,3\}}\right),\\
\Delta_3 &= p_1\left(\overline{p}_2\overline{p}_3 q_{\{1\},\{1\}}+p_2\overline{p}_3 q_{\{1\},\{1,2\}}+\overline{p}_2 p_3 q_{\{1\}|\{1,3\}}\right),\\
\Delta_4 &= p_2\left(\overline{p}_1\overline{p}_3 q_{\{2\},\{2\}}+p_1\overline{p}_3 q_{\{2\},\{1,2\}}+\overline{p}_1 p_3 q_{\{2\}|\{2,3\}}\right),\\
\Delta_5 &= p_1 p_2\overline{p}_3 q_{\{1,2\},\{1,2\}}.
\end{aligned}\tag{B.4}$$

Note that

$$G_3(0,0)=\lim_{t\to\infty}E(\mathbf{1}[N_1^t=0,\ N_2^t=0]),\tag{B.5}$$

$$G_3(x,0)-G_3(0,0)=\lim_{t\to\infty}E\left(x^{N_1^t}\mathbf{1}[N_1^t>0,\ N_2^t=0]\right),\tag{B.6}$$

$$G_3(0,y)-G_3(0,0)=\lim_{t\to\infty}E\left(y^{N_2^t}\mathbf{1}[N_1^t=0,\ N_2^t>0]\right),\tag{B.7}$$

$$\begin{aligned}
&G_3(x,y)+G_3(0,0)-G_3(x,0)-G_3(0,y)\\
&\quad=\lim_{t\to\infty}E\left(x^{N_1^t}y^{N_2^t}\mathbf{1}[N_1^t>0,\ N_2^t>0]\right).
\end{aligned}\tag{B.8}$$

Assuming the Simplistic Assumption, from (B.3), it follows that

$$\begin{aligned}
K(x,y)G_3(x,y) &= a(x,y)G_3(0,y)+b(x,y)G_3(x,0)\\
&\quad+c(x,y)G_3(0,0),
\end{aligned}\tag{B.9}$$

where $K(x,y), a(x,y), b(x,y)$, and $c(x,y)$ are given in (17).

C. Proof of Lemma 2

When $y\to 0^+$,

$$\begin{aligned}
A(0^+)&\sim\left[\lambda_1\overline{p}_1p_2\left(1-\frac{p_3}{2}\right)\right]\left(\lambda_2+\frac{\overline{\lambda}_2}{y}\right),\\
B(0^+)&\sim\left[-\lambda_1\overline{p}_1p_2\left(1-\frac{p_3}{2}\right)-\lambda_1\frac{p_1p_2\overline{p}_3}{2}\right]\left(\lambda_2+\frac{\overline{\lambda}_2}{y}\right),\\
C(0^+)&\sim\left[\overline{\lambda}_1\frac{p_1p_2\overline{p}_3}{2}\right]\left(\lambda_2+\frac{\overline{\lambda}_2}{y}\right).
\end{aligned}\tag{C.1}$$

Hence,

$$\begin{aligned}
t(0^+,\phi)&=-B(0^+)-2\cos\phi\sqrt{A(0^+)C(0^+)}\\
&<-\Bigg\{\left[\lambda_1\overline{p}_1p_2\left(1-\frac{p_3}{2}\right)+\lambda_1\frac{p_1p_2\overline{p}_3}{2}\right]\\
&\quad+2\sqrt{\left[\lambda_1\overline{p}_1p_2\left(1-\frac{p_3}{2}\right)\right]\cdot\left[\lambda_1\frac{p_1p_2\overline{p}_3}{2}\right]}\Bigg\}\\
&\quad\times\left(\lambda_2+\frac{\overline{\lambda}_2}{y}\right)\\
&=-\infty,
\end{aligned}\tag{C.2}$$

that is, $t(0^+,\phi)=-\infty$.

When $y=1$, then,

$$\begin{aligned}
A(1)&=\lambda_1\left[1-p_1\overline{p}_2\left(1-\frac{p_3}{2}\right)-\frac{p_1p_2\overline{p}_3}{2}\right],\\
B(1)&=(2\lambda_1-1)\left[p_1\overline{p}_2\left(1-\frac{p_3}{2}\right)+\frac{p_1p_2\overline{p}_3}{2}\right]-\lambda_1,\\
C(1)&=\overline{\lambda}_1\left[p_1\overline{p}_2\left(1-\frac{p_3}{2}\right)+\frac{p_1p_2\overline{p}_3}{2}\right].
\end{aligned}\tag{C.3}$$

It can be easily seen that $-B(1)=A(1)+C(1)$, hence $t(1,\phi)=-B(1)-2\cos\phi\sqrt{A(1)C(1)}>0$.

As $y\to\infty$,

$$\begin{aligned}
A(y)&\sim\lambda_1\lambda_2\Delta y,\\
B(y)&\sim\left[\lambda_1p_1\overline{p}_2\left(1-\frac{p_3}{2}\right)+\overline{\lambda}_1\Delta\right]\lambda_2y,\\
C(y)&\sim\overline{\lambda}_1\lambda_2p_1\overline{p}_2\left(1-\frac{p_3}{2}\right)y,
\end{aligned}\tag{C.4}$$

then,

$$
\begin{aligned}
t(y,\phi) &< -B(y) + 2\sqrt{A(y)C(y)} \\
&= -y\lambda_2 \left\{ \lambda_1 p_1 \overline{p}_2 \left(1 - \frac{p_3}{2}\right) + \overline{\lambda}_1 \Delta \right. \\
&\qquad \left. - 2\sqrt{\left(\overline{\lambda}_1 \Delta\right)\left[\lambda_1 p_1 \overline{p}_2 \left(1 - \frac{p_3}{2}\right)\right]} \right\} \\
&= -y\lambda_2 \left(\sqrt{\lambda_1 p_1 \overline{p}_2 \left(1 - \frac{p_3}{2}\right)} - \sqrt{\overline{\lambda}_1 \Delta} \right)^2 .
\end{aligned}
\tag{C.5}
$$

Hence, $t(\infty, \phi) < 0$.

Consequently, $t(y,\phi) = 0$ has at least two real roots $r_1(\phi)$ and $r_2(\phi)$ satisfying $0 < r_1(\phi) < 1 < r_2(\phi)$. Since $y^2 t(y,\phi)t(y,\phi+\pi)$ is a polynomial of degree four in the variable y, it can be deduced that $t(y,\phi)$ has exactly two real roots, and this completes the proof.

D. Proof of Lemma 3

The first part of the lemma results from the general theory of polynomials of two complex variables. The second assertion is proved by using Rouche's theorem as below.

$K(x,y)$ can be rewritten as

$$
K(x,y) = \frac{xy - xyF(x,y)g(x,y)}{xyF(x,y)}, \tag{D.1}
$$

where

$$
g(x,y) = \frac{p_1 \overline{p}_2 (1 - p_3/2)}{x} + \frac{p_2 \overline{p}_1 (1 - p_3/2)}{y} + \frac{p_1 p_2 \overline{p}_3}{2xy} + \Delta. \tag{D.2}
$$

For $|y| = 1$ and $y \neq 1$, $|x| = 1$,

$$
\begin{aligned}
&|xyF(x,y)g(x,y)| \\
&\quad = \left| \left(x\lambda_1 + \overline{\lambda}_1\right)\left(y\lambda_2 + \overline{\lambda}_2\right) \right. \\
&\qquad \times \left[p_1 \overline{p}_2 \left(1 - \frac{p_3}{2}\right) y + p_2 \overline{p}_1 \left(1 - \frac{p_3}{2}\right) x \right. \\
&\qquad\quad \left.\left. + \frac{p_1 p_2 \overline{p}_3}{2} + +\Delta xy \right] \right| \\
&\quad \leq 1 \\
&\quad = |xy|.
\end{aligned}
\tag{D.3}
$$

Based on Rouche's theorem, this implies that for $|y| = 1$, $y \neq 1$, there exists exactly one x, $|x| < 1$, such that $xy - xyF(x,y)g(x,y) = 0$ and hence $K(x,y) = 0$.

For $y = 1$, $K(x,1) = 0$ reduces to

$$
(x-1)\left(x - \frac{\overline{\lambda}_1 \left[p_1 \overline{p}_2 (1 - p_3/2) + p_1 p - 2\overline{p}_3/2 \right]}{\lambda \left[1 - p_1 \overline{p}_2 (1 - p_3/2) - p_1 p - 2\overline{p}_3/2 \right]} \right) = 0. \tag{D.4}
$$

When $\lambda_1 < p_1 \overline{p}_2 (1 - p_3/2) + p_1 p_2 \overline{p}_3/2$ as implied by (19), $x = 1$ is the only root.

References

[1] G. Fayolle and R. Iasnogorodski, "Two coupled processors: the reduction to a Riemann-Hilbert problem," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 47, no. 3, pp. 325–351, 1979.

[2] L. G. Roberts, *Aloha Packet System with and without Slots and Capture*, Stanford Research Institute, Advanced Research Projects Agency, Network Information Center, 1972, ASS Note 8.

[3] D. Bertsekas and R. Gallager, *Data Networks*, Prentice Hall, Upper Saddle River, NJ, USA, 2nd edition, 1992.

[4] B. S. Tsybakov and V. A. Mikhailov, "Ergodicity of slotted ALOHA system," *Problems of Information Transmission*, vol. 15, no. 4, pp. 73–87, 1979.

[5] W. Szpankowski, "Bounds for queue lengths in a contention packet broadcast system," *IEEE Transactions on Communications*, vol. 34, no. 11, pp. 1132–1140, 1986.

[6] R. R. Rao and A. Ephremides, "On the stability of interacting queues in a multiple-access system," *IEEE Transactions on Information Theory*, vol. 34, no. 5, pp. 918–930, 1988.

[7] V. Anantharam, "The stability region of the finite-user slotted ALOHA protocol," *IEEE Transactions on Information Theory*, vol. 37, no. 3, pp. 535–540, 1991.

[8] W. Szpankowski, "Stability conditions for some multiqueue distributed systems: buffered random access systems," *Advances in Applied Probability*, vol. 26, pp. 498–515, 1994.

[9] W. Luo and A. Ephremides, "Stability of N interacting queues in random-access systems," *IEEE Transactions on Information Theory*, vol. 45, no. 5, pp. 1579–1587, 1999.

[10] V. Naware, G. Mergen, and L. Tong, "Stability and delay of finite-user slotted ALOHA with multipacket reception," *IEEE Transactions on Information Theory*, vol. 51, no. 7, pp. 2636–2656, 2005.

[11] S. Ghez, S. Verdu, and S. C. Schwartz, "Stability properties of slotted ALOHA with multipacket reception capability," *IEEE Transactions on Automatic Control*, vol. 33, no. 7, pp. 640–649, 1988.

[12] S. Ghez, S. Verdu, and S. C. Schwartz, "Optimal decentralized control in the random access multipacket channel," *IEEE Transactions on Automatic Control*, vol. 34, no. 11, pp. 1153–1163, 1989.

[13] J. Luo and A. Ephremides, "On the throughput, capacity, and stability regions of random multiple access," *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2593–2607, 2006.

[14] B. Shrader and A. Ephremides, "Random access broadcast: stability and throughput analysis," *IEEE Transactions on Information Theory*, vol. 53, no. 8, pp. 2915–2921, 2007.

[15] H. Wang and T. Li, "Hybrid ALOHA: a novel MAC protocol," *IEEE Transactions on Signal Processing*, vol. 55, no. 12, pp. 5821–5832, 2007.

[16] S. Verdú, *Multiuser Detection*, Cambridge University Press, New York, NY, USA, 1998.

[17] M. L. Honig and H. V. Poor, "Adaptive interference suppression," *Wireless Communications*, pp. 64–128, 1998.

[18] R. M. Loynes, "The stability of a queue with non-independent inter-arrival and service times," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 58, pp. 497–520, 1962.

[19] P. Nain, "Analysis of a two node ALOHA network with infinite capacity buffers," in *Proceedings of the International Seminar on Computer Networking*, pp. 2.2.1–2.2.16, Tokyo, Japan, 1985.

[20] N. I. Mushkelishvili, *Singular Integral Equations*, Noordhoff, Groningen, The Netherlands, 1953.

[21] J. W. Cohen and O. J. Boxma, *Boundary Value Problems in Queueing System Analysis*, vol. 79 of *North-Holland Mathematics Studies*, North-Holland, New York, NY, USA, 1983.