Hindawi Publishing Corporation
EURASIP Journal on Wireless Communications and Networking
Volume 2010, Article ID 921427, 13 pages
doi:10.1155/2010/921427

Research Article

Design Criteria for Hierarchical Exclusive Code with Parameter-Invariant Decision Regions for Wireless 2-Way Relay Channel

Tomas Uricar and Jan Sykora

Department of Radio Engineering, Faculty of Electrical Engineering, Czech Technical University in Prague, Technicka 2, 166 27 Praha 6, Czech Republic

Correspondence should be addressed to Tomas Uricar, uricatom@fel.cvut.cz

Received 31 December 2009; Revised 12 May 2010; Accepted 30 June 2010

Academic Editor: Meixia Tao

The unavoidable parametrization of the wireless link represents a major problem of the network-coded modulation synthesis in a 2-way relay channel. Composite (hierarchical) codeword received at the relay is generally parametrized by the channel gain, forcing any processing on the relay to be dependent on channel parameters. In this paper, we introduce the codebook design criteria, which ensure that all permissible hierarchical codewords have decision regions invariant to the channel parameters (as seen by the relay). We utilize the criterion for parameter-invariant constellation space boundary to obtain the codebooks with channel parameter-invariant decision regions at the relay. Since the requirements on such codebooks are relatively strict, the construction of higher-order codebooks will require a slightly simplified design criteria. We will show that the construction algorithm based on these relaxed criteria provides a feasible way to the design of codebooks with arbitrary cardinality. The promising performance benefits of the example codebooks (compared to a classical linear modulation alphabets) will be exemplified on the minimum distance analysis.

1. Introduction

The physical-layer coding in the wireless multinode and multisource scenarios is currently under a heavy investigation in the research community. The cooperative relaying scenarios for two-way communication (see, e.g., [1, 2]), and particularly the scenarios based on the principles similar to Network Coding (NC) [3], are foreseen to have a great potential even for the wireless communication networks. Although the pure NC operates with a discrete (typical binary) alphabet over lossless discrete channels, its principles can be extended into the wireless domain. Such an extension is however nontrivial, because the signal space link models (e.g., MAC phase in relay communications) lack a simple finite field properties as found and used in a pure discrete NC. There are still only very limited results available even for the simplest possible scenarios like 2-Way Relay Channel (2-WRC). A brief overview of the current state of art of the bidirectional relaying scenarios is available in [4] and in references therein.

A complete revamp of the physical-layer modulation/coding, respecting inherently and from the beginning the structure of the multinode network with possible multiple sources of information, is foreseen to be a preferred solution [5]. This principle is sometimes called Wireless Network Coding (WNC) or Physical Network Coding (PNC). However, we believe that the term Network Coded Modulation (NCM) better describes the phenomenon of the modulation/coding aware of the surrounding network structure. The major benefits of these communication principles are given by the possibility to increase a throughput in a MAC phase of the bidirectional communication, and by the inherently increased reliability of the BC stage [6].

The strategy where the relay decodes only a hierarchical codeword is called by some authors Denoising (DNF Strategy) [7]; however the term "denoising" seems to be rather connected to the symbol level treatment (as done in [7]). We feel that a more generic term Hierarchical Decoding and Forward (HDF) [8, 9] is better suited for possible application on more complicated codebooks and channel structures. Increased MAC phase throughput of DNF (HDF) strategy provides performance improvement against the standard techniques based on Amplify & Forward or Joint Decode & Forward (e.g., [10]) paradigms. In the MAC phase of the DNF strategy relay "denoises" the received signal, which means that it performs decisions directly on the superimposed symbols without actually distinguishing the individual symbols from both sources. Together with the eXclusive law [7] applied on the relay output, symbol mapping makes it possible to get joint throughput gains similar to the discrete NC case.

The paper in [9] introduces Hierarchical eXclusive Code (HXC) layered design which relies on a concatenation of the exclusive alphabet and outer standard capacity-approaching code. Lattice-based code construction [11, 12], using the principles from [13] is limited to the nonparametric Gaussian channels. Authors of [11] present the simplest realization of HDF strategy with minimal cardinality mapping, which they call "modulo decoding". More general relay output mapping, which considers also the possibility of extended cardinality, is introduced in [8].

The channel parametrization proved to be a major problem of synthesizing relay WNC in Denoise and Forward (DNF) strategies proposed in [6, 7, 14]. Specific channel parametrization can invoke the eXclusive law [7] failures, resulting in significant performance degradation (see e.g., [7, 9]). The authors of [6] propose two adaptive solutions to overcome this problem. The first approach prerotates the transmitted signal (closed loop adaptation required) in such a way that the constellation observed at the relay is invariant to the channel parameter. The second solution uses an adaptive relay decision DNF maps, choosing the optimal one for a given parametrization. The particular map index needs to be passed along with the data message at the broadcast phase.

The adaptive solutions are generally not well suited for fast-fading channels. Moreover, the increased BC phase overhead (e.g., larger adaptive DNF map set, increased cardinality of the relay output) of these adaptive solutions was observed for higher-order modulations (e.g., 16-QAM) [6].

This paper approaches the problems of the MAC phase channel parametrization in the HDF relaying from a different angle. We design the alphabets (codewords) used by source nodes A and B in such a way that the resulting hierarchical codeword visible at the relay has channel parameter-*invariant* decision regions. The design criteria for a Parametric Hierarchical eXclusive Code (PHXC), which satisfies the requirement of parameter-invariant decision regions at the relay, are presented in the form of required conditions for PHXC hierarchical codeword pairs in [5]. The fulfillment of these design criteria force the particular constellation space boundary (given by the set of points which have an identical Euclidean distance to the both corresponding hierarchical codewords) to be invariant to the channel parametrization.

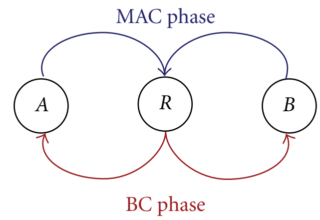


Figure 1: Model of 2-WRC in half-duplex mode.

Complete individual codebooks could be designed to be pairwise parameter-invariant only for those pairs of hierarchical codewords whose decision regions at the relay are mutually neigbouring and which fall into two distinct mapping regions of the relay output. The PHXC design criteria then guarantee the *parameter-invariance* of the corresponding *pairwise* (decision) *boundaries*.

Another way of how to synthesize *complete PHXC codebook* is to apply the pairwise PHXC design criteria on *all "critical" hierarchical codeword pairs*, that is, on all hierarchical codeword pairs which must always obey the exclusive law. Such an approach will force all corresponding pairwise boundaries (i.e., not only the ones which are directly affecting the decision regions shape) to be parameter-invariant. Although this requirement could be relatively strict, it allows us to express the codebook design criteria in a compact set of required conditions. In this paper, we present the *design criteria* for the complete parametric hierarchical exclusive codebook and show that all requirements of the extended design criteria can be satisfied at once only if the terminals use different individual codebooks.

2. System Model and Definitions

We adopt the system model presented in [5]. A 2-WRC working in a half-duplex mode (nodes cannot simultaneously receive and transmit) is assumed. The end-nodes of the system are denoted as A and B, and the relay is denoted as R (Figure 1).

2.1. MAC Phase. The constellation space signal received at R in MAC phase is

$$\mathbf{x} = h_A \mathbf{s}_A + h_B \mathbf{s}_B + \mathbf{w}, \tag{1}$$

where $\mathbf{w}$ is AWGN, h_A, h_B are scalar complex channel coefficients (constant during the observation and known at R), and $\mathbf{s}_A$, $\mathbf{s}_B$ are transmitted signal space codewords. The useful signal $(h_A \mathbf{s}_A + h_B \mathbf{s}_B)$ can be equivalently expressed (after a rescaling by $1/h_A$) as

$$\mathbf{u} = \mathbf{s}_A + \alpha \mathbf{s}_B, \tag{2}$$

where $\alpha = h_B/h_A$. The only purpose of this "rescaling" is an attempt to simplify the signal analysis in parametric MAC channel by introducing a useful signal model which incorporates the influence of both channel parameters

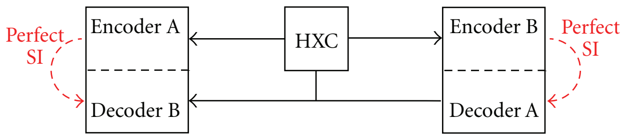


Figure 2: 2-WRC with HXC and perfect Side Information.

(h_A, h_B) in the single one parameter (α). The equivalent MAC channel model is in this case given by

$$\mathbf{x} = h_A \mathbf{u} + \mathbf{w}. \tag{3}$$

The signal space codewords $\mathbf{s}_i$ $(i \in \{A, B\})$ are drawn from individual codebooks $\mathcal{B}_A$, $\mathcal{B}_B$ (individual codebooks can be different in general). The equivalent hierarchical codewords $\mathbf{u} \in \mathcal{B}_u(\alpha)$, as seen by R, have generally the codebook parametrized by α. The number of individual codewords in $\mathcal{B}_A$ and $\mathcal{B}_B$ is assumed to be equal, that is, $|\mathcal{B}_A| = |\mathcal{B}_B| = N$, and the number of hierarchical codewords is $|\mathcal{B}_u(\alpha)| \leq N^2$. Throughout this paper, we assume only a *minimal* cardinality of the hierarchical codebook $|\mathcal{B}_u(\alpha)| = N$. For a general discussion on hierarchical codebook cardinality see [9].

2.2. BC Phase. The relay R re-encodes the codeword $\mathbf{u}$ into $\mathbf{s}_R = \mathbf{s}_R(\mathbf{u}) \in \mathcal{B}_R$ and sends it during the BC phase. Notice that the relay decodes only a *hierarchical* codeword $\mathbf{u}$ and *not* the *individual* codewords $\mathbf{s}_A, \mathbf{s}_B$. This corresponds to the DNF (HDF) relaying strategy. In the BC phase, the nodes receive

$$\mathbf{x}_i = h_{Ri} \mathbf{s}_R + \mathbf{w}_i, \tag{4}$$

where $i \in \{A, B\}$, h_{Ri} are channel complex gains, and $\mathbf{w}_i$ is AWGN.

3. Parametric Hierarchical Exclusive Code in 2-WRC

3.1. HDF Strategy in Parametric 2-WRC. The joint 2-source signal space codebook is called eXclusive Code (XC) $\mathcal{C}(\mathbf{s}_A, \mathbf{s}_B) \in \mathcal{B}_R$ if and only if the exclusive law [7] of the network coding holds

$$\begin{aligned} \mathcal{C}(\mathbf{s}_A, \mathbf{s}_B) \neq \mathcal{C}(\mathbf{s}'_A, \mathbf{s}_B) &\iff \mathbf{s}_A \neq \mathbf{s}'_A, \\ \mathcal{C}(\mathbf{s}_A, \mathbf{s}_B) \neq \mathcal{C}(\mathbf{s}_A, \mathbf{s}'_B) &\iff \mathbf{s}_B \neq \mathbf{s}'_B. \end{aligned} \tag{5}$$

Assuming that the receiver has perfect a priori Side Information (SI) on its own data, the decoding of the XC-encoded source A data is not affected by the source B data (and vice-versa) (Figure 2). The capacity region has a *rectangular* shape which can be outside the 2-user MAC region of traditional Decode and Forward (DF) strategy [4].

The relay processing in the HDF strategy consists generally of the decoding function $\hat{\mathbf{u}} = \mathcal{D}_R(\mathbf{x})$ and the encoding function $\mathbf{s}_R = \mathcal{C}_R(\hat{\mathbf{u}})$. If the hierarchical data rate R_u is below the equivalent hierarchical MAC channel (3) capacity, then the HDF Decoder (HDFD) can provide perfect decisions $\hat{\mathbf{u}} = \mathbf{u}$ and the HDF design reduces to the design of the HDF Coder (HDFC) function $\mathcal{C}_R(\cdot)$ such that

$$\mathcal{C}_R(\mathbf{u}(\mathbf{s}_A, \mathbf{s}_B)) = \mathcal{C}(\mathbf{s}_A, \mathbf{s}_B), \tag{6}$$

where $\mathcal{C}(\mathbf{s}_A, \mathbf{s}_B)$ is XC. Such a code $\mathcal{C}_R(\cdot)$ will be called Hierarchical eXclusive Code (HXC).

A major problem occurs when we apply the HDF strategy to the wireless constellation space *parametric* channels. The constellation space model of the MAC phase is continuously valued (3), and hence it lacks a simple finite field properties (as found and used in a pure discrete NC). The codewords visible at the relay are parametrized $\mathbf{u}(\alpha) \in \mathcal{B}_u(\alpha)$ and the decision regions of the HDF re-encoder generally depend on α. A structure of the processing at the relay is shown in Figure 3.

The exclusive law in *parametric* channel [5] implies

$$\begin{aligned} \mathbf{u}(\mathbf{s}_A, \mathbf{s}_B, \alpha) \neq \mathbf{u}(\mathbf{s}'_A, \mathbf{s}_B, \alpha) &\iff \mathbf{s}_A \neq \mathbf{s}'_A, \\ \mathbf{u}(\mathbf{s}_A, \mathbf{s}_B, \alpha) \neq \mathbf{u}(\mathbf{s}_A, \mathbf{s}'_B, \alpha) &\iff \mathbf{s}_B \neq \mathbf{s}'_B \end{aligned} \tag{7}$$

for all α. The decoding and encoding functions generally depend on channel parameters $\mathcal{D}_{R(\alpha)}(\cdot)$, $\mathcal{C}_{R(\alpha)}(\cdot)$. The code which has the HDF functions $\mathcal{D}_R(\cdot)$ and $\mathcal{C}_R(\cdot)$ invariant to the channel parametrization (α) is called Parametric Hierarchical eXclusive Code (PHXC) [5]. Generally the PHXC comprises codebooks $\mathcal{B}_A$, $\mathcal{B}_B$, $\mathcal{B}_R$ and the re-encoding functions $\mathcal{D}_{R(\alpha)}(\cdot)$ and $\mathcal{C}_{R(\alpha)}(\cdot)$. One of the possible ways of how to design the PHXC is to design the codebooks $\mathcal{B}_A$ and $\mathcal{B}_B$ in such a way that the decoding function $\mathcal{D}_{R(\alpha)}(\cdot)$ does not depend on α, that is, the HDFD decision regions are *parameter-invariant* [5].

The codebook design for parametric channels can generally focus on the two different design goals. One goal is the *parameter-invariant structure* of the relay processing, which can be achieved if the codebook design forces the decision regions at the relay to be independent on the actual channel parameter values. The second goal is the *parameter-invariant performance* of the entire system, that is, the codebook design with performance (e.g., rate) resistant to the channel parametrization. This paper mainly addresses the first goal, the codebook design criteria for parameter-invariant decoder structure. As we will show in the later sections, the "reduced" version of the proposed codebook design criterion shows also some promising (parameter-invariant) performance results.

3.2. HDF Decoder Decision Regions. We denote the codewords in codebooks as follows, $\mathcal{B}_A = \{\mathbf{s}^{i_A}\}_{i_A}$, $\mathcal{B}_B = \{\mathbf{s}^{i_B}\}_{i_B}$ and $\mathcal{B}_u = \{\mathbf{u}^k\}_k$. Let $\mathbf{u}^{k(i_A,i_B)}(\alpha) = \mathbf{s}^{i_A} + \alpha \mathbf{s}^{i_B}$ be the equivalent hierarchical codeword received at the relay. Codeword indices k, i_A, i_B must obviously obey the exclusive law (7). Note that the index of the hierarchical codeword k is a function of the pair of individual codeword indices (i_A, i_B), hence it is useful to list all permissible combinations of individual codewords $\mathbf{s}^{i_A}$, $\mathbf{s}^{i_B}$ (and corresponding hierarchical codewords $\mathbf{u}^{k(i_A,i_B)}$) in a "hierarchical codeword table" (Table 1). We generally assume that all codebooks are subsets of 2-dimensional vector space over the field $\mathbb{F}$ $(\mathcal{B}_A, \mathcal{B}_B, \mathcal{B}_u, \mathcal{B}_R \subset \mathbb{F}^2)$ and that the parameter is a scalar in $\mathbb{F}$, $\alpha \in \mathbb{F}$. The field is typically the set of real or complex numbers.

Definition 1. A pairwise boundary $\mathcal{R}^{kl}(\alpha)$ is the set of points having the same (constellation space) Euclidean

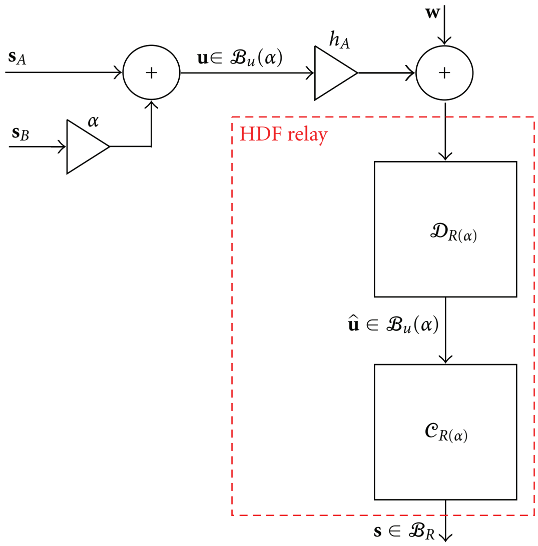


Figure 3: Equivalent model of HDF strategy relay processing in parametric channel.

Table 1: Example of hierarchical codeword table ($|\mathcal{B}_A| = |\mathcal{B}_B| = N$).

	i_{B1}	i_{B2}	$\cdots$	i_{BN}
i_{A1}	$\mathbf{u}^{(i_{A1},i_{B1})}$	$\mathbf{u}^{(i_{A1},i_{B2})}$	$\cdots$	$\mathbf{u}^{(i_{A1},i_{BN})}$
i_{A2}	$\mathbf{u}^{(i_{A2},i_{B1})}$	$\mathbf{u}^{(i_{A2},i_{B2})}$	$\cdots$	$\mathbf{u}^{(i_{A2},i_{BN})}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
i_{AN}	$\mathbf{u}^{(i_{AN},i_{B1})}$	$\mathbf{u}^{(i_{AN},i_{B2})}$	$\cdots$	$\mathbf{u}^{(i_{AN},i_{BN})}$

distance from a pair of hierarchical codewords $\mathbf{u}^{k(i_A,i_B)}(\alpha)$ and $\mathbf{u}^{l(i'_A,i'_B)}(\alpha)$ for any $k \neq l$. A *pairwise boundaries* set $\mathcal{S}_{PB}$ is the union of all pairwise boundaries $\mathcal{R}^{kl}(\alpha)$.

A pairwise boundary (see the example in Figure 4) is defined for every permissible pair of hierarchical codewords $(\mathbf{u}^k(\alpha),\mathbf{u}^l(\alpha))$. From the perspective of the codebook design, the most critical are those pairs of hierarchical codewords, which have one of the comprising individual codewords identical ($\mathbf{s}_A = \mathbf{s}'_A$ or $\mathbf{s}_B = \mathbf{s}'_B$). These hierarchical codeword pairs may directly violate the exclusive law (7), if some specific value of parametrization cause them to fall into an identical decision region of the relay decoder. The codewords from such pair must hence be designed appropriately to ensure that they always fall into two distinct mapping regions of the output HDF codebook $\mathcal{B}_R$, otherwise the errorless communication would be impossible. Pairwise boundaries between all such pairs of hierarchical codewords constitute some subset of $\mathcal{S}_{\text{PB}}$, as it is obvious from the following definition.

Definition 2. A *critical boundaries subset* $\mathcal{S}_{\text{CB}} \subset \mathcal{S}_{PB}$ is the set of all pairwise boundaries $\mathcal{R}^{kl}(\alpha)$ between all permissible hierarchical codewords pairs $\mathbf{u}^{k(i_A,i_B)}(\alpha)$, $\mathbf{u}^{l(i'_A,i'_B)}(\alpha)$ which have $i_A = i'_A$ or $i_B = i'_B$.

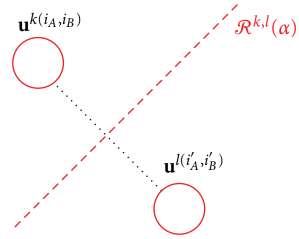


Figure 4: Visualization of the pairwise boundary in the constellation space.

Pairwise boundary $\mathcal{R}^{kl}$ between the hierarchical codewords pair $\mathbf{u}^{k(i_A,i_B)}(\alpha)$, $\mathbf{u}^{l(i'_A,i'_B)}(\alpha)$ is hence classified as critical ($\mathcal{R}^{kl}_{\text{CB}}$) by Definition 2, if the corresponding hierarchical codewords reside in the same row ($i_A = i'_A$) or column ($i_B = i'_B$) of the hierarchical codeword table (Table 1).

3.3. Pairwise PHXC Design Criteria. As mentioned above, one of the possible ways of how to design the PHXC is to design the codebooks $\mathcal{B}_A$ and $\mathcal{B}_B$ in such a way that the decoding function $\mathcal{D}_{R(\alpha)}(\cdot)$ would not depend on α, that is, the HDFD decision regions are *α-invariant*. The shape of the HDFD decision regions is always given directly by some subset of pairwise boundaries, which we will call the *active boundaries subset* ($\mathcal{S}_{\text{AB}} \subset \mathcal{S}_{\text{PB}}$)—see the example in Figure 5. In general, the active boundaries subset $\mathcal{S}_{\text{AB}}$ does not have to comprise solely the boundaries from $\mathcal{S}_{\text{CB}}$ ($\mathcal{S}_{\text{AB}} \not\subseteq \mathcal{S}_{\text{CB}}$). As it is also obvious from Figure 5, the final shape of the HDFD decision regions generally does not have to be formed by all boundaries from $\mathcal{S}_{\text{CB}}$. Boundaries for some index pairs could be overlapped by other decision boundaries. For example, boundaries between two neigbouring hierarchical codewords (in one column or row of the hierarchical codeword table) do not have to appear as a true decision boundaries of the overall hierarchical codebook. However, considering all, even the "masked" ones, enables simplified parametric codebook construction at the expense of fulfilling stricter criterion than actually required. Such code design rules are thus sufficient but not necessary ones.

The pairwise design criterion for the *α-invariant pairwise boundary* $\mathcal{R}^{kl}$ (i.e., for the α-invariant hierarchical codeword pair $\mathbf{u}^{k(i_A,i_B)}$ and $\mathbf{u}^{l(i'_A,i'_B)}$) in $\mathbb{F}^2$ is (under some limitations) derived as a pair of required conditions in [5]:

$$\left\langle \mathbf{s}^{i_A} - \mathbf{s}^{i'_A}; \mathbf{s}^{i_B} + \mathbf{s}^{i'_B} \right\rangle = 0, \tag{8}$$

$$\left\langle \mathbf{s}^{i_B} - \mathbf{s}^{i'_B}; \mathbf{s}^{i_B} + \mathbf{s}^{i'_B} \right\rangle = 0. \tag{9}$$

4. Design Criteria for Complete PHXC Codebooks

The *final shape* of the HDFD decision regions is given entirely by active boundaries ($\mathcal{R}^{kl}_{\text{AB}}(\alpha) \in \mathcal{S}_{\text{AB}}$). Hence, it could seem quite reasonable to apply the pairwise design criteria (8), and (9) just to these boundaries in $\mathcal{S}_{\text{AB}}$. Note that in this case the design criteria would ensure that the constellation space "position" of all boundaries from $\mathcal{S}_{\text{AB}}$ will remain fixed,

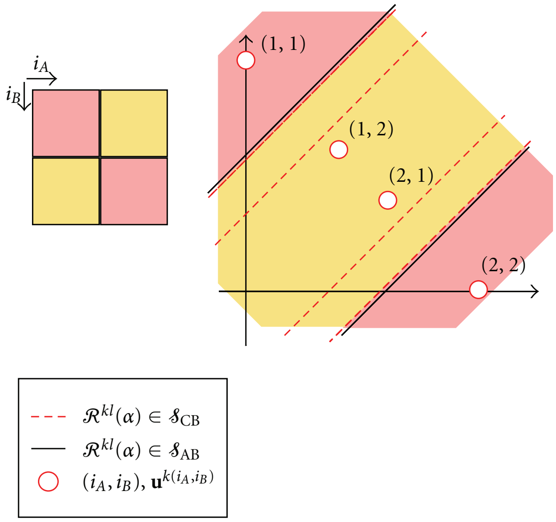


FIGURE 5: HDFD decision regions' shape example (real-valued 2-dimensional example codebook). Note that some boundaries lie *inside* the decision region corresponding to *one hierarchical codeword* (given by the same region colour). Such boundaries *do not* affect the final decision region's shape, and hence can be considered as "masked".

however some other boundaries could potentially "move" along with the varying channel parameter α.

This "boundary movement" could (for some values of α) change the HDFD decision regions shape and hence break the requirement of parameter-invariant HDFD decision regions (Figure 6). One way of how to potentially avoid this undesirable behavior is to apply the design criteria on *all critical boundaries* ($\mathcal{R}^{kl}_{\mathrm{CB}}(\alpha) \in \mathcal{S}_{\mathrm{CB}}$), thus requiring all pairwise boundaries in $\mathcal{S}_{\mathrm{CB}}$ to be α-invariant. As we will prove later in this section, this condition will be sufficient to force even the entire set $\mathcal{S}_{\mathrm{PB}}$ to be parameter-invariant.

4.1. E-PHXC Design Criteria. Forcing all critical pairwise boundaries to be α-invariant could be a relatively strict requirement; nevertheless it allows us to express the design criteria in a compact set of required conditions, and it avoids the movement of all critical boundaries (complete set $\mathcal{S}_{\mathrm{CB}}$), which are dominantly responsible for the final shape of the HDFD decision regions. We apply the design criteria (8), and (9) for the parameter-invariant pairwise boundary to *all critical boundaries;* hence the *extended* design criteria for a *complete PHXC codebooks* will be derived.

A code which has all the *critical* boundaries ($\mathcal{R}^{kl}_{\mathrm{CB}}(\alpha) \in \mathcal{S}_{\mathrm{CB}}$) invariant to the channel parameter will be called Extended Parametric Hierarchical eXclusive Code (E-PHXC). Now we will formally define the E-PHXC codebooks and introduce the necessary conditions for the codebooks' design in Lemma 4 (proof is available in Appendix A).

Definition 3. The codebooks $\mathcal{B}_A = \{\mathbf{s}^{i_A}\}_{i_A}$, $\mathcal{B}_B = \{\mathbf{s}^{i_B}\}_{i_B}$ are the E-PHXC when all the critical boundaries $\mathcal{R}^{kl}_{\mathrm{CB}}(\alpha) \in \mathcal{S}_{\mathrm{CB}}$ for hierarchical codebook $\mathcal{B}_u(\alpha)$ at the relay are α-invariant.

Lemma 4. *The codebooks $\mathcal{B}_A = \{\mathbf{s}^{i_A}\}_{i_A}$, $\mathcal{B}_B = \{\mathbf{s}^{i_B}\}_{i_B}$ are the E-PHXC if the following conditions hold:*

$$\left\langle \mathbf{s}^{i_A} - \mathbf{s}^{i'_A}; \mathbf{s}^{i_B} \right\rangle = 0 \quad \forall i_A < i'_A, \tag{10}$$

$$\left\langle \mathbf{s}^{i_B} - \mathbf{s}^{i'_B}; \mathbf{s}^{i_B} + \mathbf{s}^{i'_B} \right\rangle = 0 \quad \forall i_B < i'_B, \tag{11}$$

for all $i_A, i_B, i'_A, i'_B \in \{1, 2, \ldots, N\}$, where $N = |\mathcal{B}_A| = |\mathcal{B}_B|$.

4.2. E-PHXC Decoder Decision Regions. Design criteria for E-PHXC codebooks (10) and (11) force all critical boundaries (set $\mathcal{S}_{\mathrm{CB}}$) to be invariant to the channel parameter. Hence, all pairs of hierarchical codewords which are in the same row (or column) of the hierarchical codeword table (Table 1) have the corresponding pairwise boundary invariant to the channel parameter. Moreover, the design criteria are sufficient to force the *entire set* of pairwise boundaries ($\mathcal{S}_{\mathrm{PB}}$) to be parameter-invariant, that is, the constellation space boundary $\mathcal{R}^{k,l}$ between any permissible pair of hierarchical codewords is forced to be parameter-invariant by the E-PHXC design criteria (10) and (11). We will prove this in the following Lemma (proof is available in Appendix B).

Lemma 5. *If the codebook fulfills E-PHXC design criteria then it has all permissible pairwise boundaries ($\mathcal{R}^{k,l} \in \mathcal{S}_{\mathrm{PB}}$) invariant to the channel parameter.*

4.3. E-PHXC with Identical Individual Codebooks. Now we analyze the design criteria for the special case of *identical* individual codebooks $\mathcal{B}_A = \mathcal{B}_B = \mathcal{B}$. Note that by "identical codebooks" we mean codebooks which have all codewords completely identical (i.e., including the indexing of codewords in the codebook). Hence e.g. two mutually rotated BPSKs are not considered as identical. In this case, both codebooks contain the same codewords, so we may omit the subscript (A, B) from indices.

Theorem 6 (E-PHXC with identical codebooks). *The codebook $\mathcal{B} = \{\mathbf{s}^i\}_i$ is the E-PHXC if the following conditions hold:*

$$\begin{aligned} \left\| \mathbf{s}^i \right\| &= \left\| \mathbf{s}^{i'} \right\| \quad \forall i < i', \\ \left\| \mathbf{s}^1 \right\|^2 &= \left\langle \mathbf{s}^i; \mathbf{s}^{i'} \right\rangle \quad \forall i < i', \end{aligned} \tag{12}$$

for all $i, i' \in \{1, 2, \ldots N\}$, where $N = |\mathcal{B}|$.

Proof. We start with (11) from which we get for two pairs of codeword indices (i, j) and (i', j')

$$\begin{aligned} \left\langle \mathbf{s}^i - \mathbf{s}^{i'}; \mathbf{s}^i + \mathbf{s}^{i'} \right\rangle &= 0, \\ \left\| \mathbf{s}^i \right\|^2 - \left\| \mathbf{s}^{i'} \right\|^2 + j2\Im\left\{ \left\langle \mathbf{s}^i; \mathbf{s}^{i'} \right\rangle \right\} &= 0 \quad \forall i < i', \end{aligned} \tag{13}$$

where j is an imaginary unit. Should this hold for all $i < i'$, the inner products $\langle \mathbf{s}^i; \mathbf{s}^{i'} \rangle$ must be real-valued and all norms $\|\mathbf{s}^i\|, \|\mathbf{s}^{i'}\|$ must have same magnitude. Thus, the condition (11) is equivalent with conditions $\langle \mathbf{s}^i; \mathbf{s}^{i'} \rangle \in \mathbb{R}$ and $\|\mathbf{s}^i\| =$ const.

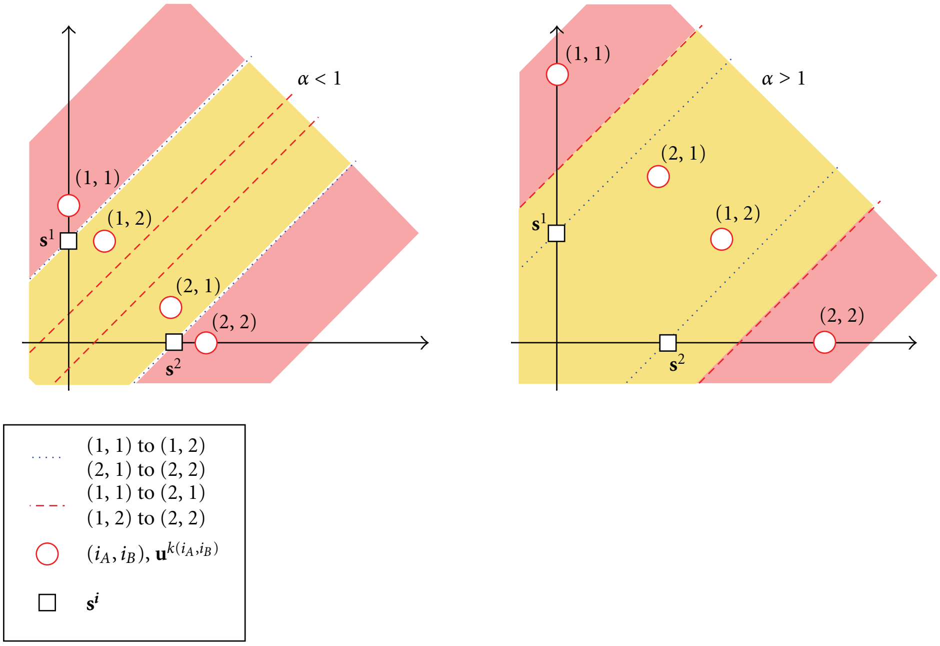


Figure 6: Movement of pairwise boundaries affects the HDFD decision regions' shape (real-valued 2-dimensional example codebook).

From (10) we get

$$\begin{gathered}\left\langle \mathbf{s}^i - \mathbf{s}^{i'}; \mathbf{s}^j \right\rangle = 0, \\ \left\langle \mathbf{s}^i; \mathbf{s}^j \right\rangle = \left\langle \mathbf{s}^{i'}; \mathbf{s}^j \right\rangle \quad \forall i < i',\end{gathered} \tag{14}$$

for all $i, i', j \in \{1, 2, \ldots N\}$. Considering the symmetry, this is equivalent to

$$\left\langle \mathbf{s}^i; \mathbf{s}^j \right\rangle = \left\langle \mathbf{s}^{i'}; \mathbf{s}^j \right\rangle \quad \forall i, i', j, \tag{15}$$

which is in turn equivalent to

$$\left\langle \mathbf{s}^i; \mathbf{s}^{i'} \right\rangle = \text{const} = \|\mathbf{s}^1\|^2, \quad \forall i, i'. \tag{16}$$

Thus the condition (10) is equivalent to $\langle \mathbf{s}^i; \mathbf{s}^{i'} \rangle = \|\mathbf{s}^1\|^2$. □

Theorem 7. *E-PHXC does not exist for any identical individual binary codebooks ($\mathcal{B}_A = \mathcal{B}_B = \mathcal{B}$, $|\mathcal{B}| = 2$).*

Proof. The binary codebook contains two individual codewords $\mathcal{B} = \{\mathbf{s}^1, \mathbf{s}^2\}$. Each codeword is a 2-dimensional vector over the field $\mathbb{F}$. Design criteria for the E-PHXC with identical binary codebooks require (from (12))

$$\|\mathbf{s}^1\| = \|\mathbf{s}^2\|, \tag{17}$$

$$\|\mathbf{s}^1\|^2 = \langle \mathbf{s}^1; \mathbf{s}^2 \rangle. \tag{18}$$

We assume that there exists $\mathbf{s}^1 \neq \mathbf{s}^2$ such that both conditions are satisfied.

The *Cauchy-Bunyakovskii-Schwartz inequality* (CBS) [15] states that for all vectors $\mathbf{x}, \mathbf{y}$

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|, \tag{19}$$

where the equality is achieved if and only if $\mathbf{x} = \gamma \mathbf{y}$ for $\gamma = \langle \mathbf{x}, \mathbf{y} \rangle / \|\mathbf{x}\|^2$. The inner product $\langle \mathbf{s}^1; \mathbf{s}^2 \rangle$ must be positive and real valued (from (18)), so $|\langle \mathbf{s}^1; \mathbf{s}^2 \rangle| = \langle \mathbf{s}^1; \mathbf{s}^2 \rangle$. Now, we apply the CBS inequality (19) on vectors $\mathbf{s}^1, \mathbf{s}^2$:

$$\begin{gathered}|\langle \mathbf{s}^1; \mathbf{s}^2 \rangle| \leq \|\mathbf{s}^1\| \cdot \|\mathbf{s}^2\|, \\ \langle \mathbf{s}^1; \mathbf{s}^2 \rangle \leq \|\mathbf{s}^1\|^2,\end{gathered} \tag{20}$$

because $\|\mathbf{s}^1\| = \|\mathbf{s}^2\|$ (from (17)). Condition (18) requires the equality in (20). This equality is achieved if and only if $\mathbf{s}^1 = \gamma \mathbf{s}^2$, where $\gamma = \langle \mathbf{s}^1; \mathbf{s}^2 \rangle / \|\mathbf{s}^1\|^2 = 1$, that is, the equality is achieved if and only if $\mathbf{s}^1 = \mathbf{s}^2$, which is a contradiction with the assumption $\mathbf{s}^1 \neq \mathbf{s}^2$. □

Corollary 8. *E-PHXC does not exist for any identical individual codebooks ($\mathcal{B}_A = \mathcal{B}_B = \mathcal{B}$, $|\mathcal{B}| = N$).*

Proof. The conditions (17) and (18) form a subset of all required conditions for any individual codebook with cardinality greater than two ($|\mathcal{B}| > 2$). As shown in a proof of Theorem 7, it is impossible to find two different codewords satisfying this condition. □

4.4. E-PHXC with Different Individual Codebooks. We proved that the individual codebook satisfying all the required design criteria does not exist if we request both codebooks to be identical. In this section, we derive the E-PHXC design criteria for the assumption of two nonidentical individual codebooks ($\mathcal{B}_A \neq \mathcal{B}_B$).

Theorem 9 (E-PHXC with different codebooks). *The codebooks $\mathcal{B}_A = \{\mathbf{s}^{i_A}\}_{i_A}$, $\mathcal{B}_B = \{\mathbf{s}^{i_B}\}_{i_B}$ are the E-PHXC if the following conditions hold:*

$$\left\|\mathbf{s}^{i_B}\right\| = \left\|\mathbf{s}^{i'_B}\right\| \quad \forall i_B < i'_B, \tag{21}$$

$$\Im\left\langle \mathbf{s}^{i_B}; \mathbf{s}^{i'_B} \right\rangle = 0 \quad \forall i_B < i'_B, \tag{22}$$

$$\left\langle \mathbf{s}^{i_A} - \mathbf{s}^{i'_A}; \mathbf{s}^{j_B} \right\rangle = 0 \quad \forall i_A < i'_A, \tag{23}$$

for all $i_A, i'_A, i_B, i'_B, j_B \in \{1, 2, \ldots, N\}$.

Proof. We start again with (11), from which we get

$$\begin{gathered} \left\langle \mathbf{s}^{i_B} - \mathbf{s}^{i'_B}; \mathbf{s}^{i_B} + \mathbf{s}^{i'_B} \right\rangle = 0, \\ \left\|\mathbf{s}^{i_B}\right\|^2 - \left\|\mathbf{s}^{i'_B}\right\|^2 + j2\Im\left\{\left\langle \mathbf{s}^{i_B}; \mathbf{s}^{i'_B} \right\rangle\right\} = 0 \quad \forall i_B < i'_B, \end{gathered} \tag{24}$$

for all $i_B, i'_B \in \{1, 2, \ldots, N\}$, which gives us directly (21) and (22). From (10) we immediately get the last condition (23). □

4.5. Example Binary Alphabet Construction Algorithm. We have shown in previous sections that E-PHXC codebooks have all pairwise boundaries invariant to the channel parameter and that they could be designed only if the sources use two different individual codebooks ($\mathcal{B}_A \neq \mathcal{B}_B$). Here we exemplify the E-PHXC design criteria for this case ((21), (22), (23)) on a few simple cases.

Assume $\mathbb{F} = \mathbb{C}, n = 2$ and two different binary codebooks $|\mathcal{B}_A| = |\mathcal{B}_B| = 2$ with code indices $i_A, i_B \in \{1, 2\}$. Value α is a complex-valued scalar. Considering these assumptions, the design criteria for a binary E-PHXC (from Theorem 9) are

$$||\mathbf{s}^{1_B}|| = ||\mathbf{s}^{2_B}||, \tag{25}$$

$$\Im\{\langle \mathbf{s}^{1_B}; \mathbf{s}^{2_B} \rangle\} = 0, \tag{26}$$

$$\langle \mathbf{s}^{1_A} - \mathbf{s}^{2_A}; \mathbf{s}^{1_B} \rangle = 0, \tag{27}$$

$$\langle \mathbf{s}^{1_A} - \mathbf{s}^{2_A}; \mathbf{s}^{2_B} \rangle = 0. \tag{28}$$

As it is obvious from (27) and (28), a trivial example of E-PHXC are codebooks $\mathcal{B}_A, \mathcal{B}_B$ with mutually *orthogonal* codewords ($\langle \mathbf{s}^{i_A}; \mathbf{s}^{i_B} \rangle = 0$ for all $\mathbf{s}^{i_A}, \mathbf{s}^{i_B}$), provided that also (25) and (26) are not violated. Some examples of these "orthogonal" binary codebooks are presented in Table 2. Codebooks $\mathcal{B}_A, \mathcal{B}_B$ spanning mutually orthogonal subspaces have additional advantage of providing unitary parameter-invariant performance (e.g., the phase rotation). The decision subspaces for both source codebooks are independent (orthogonal) and thus a unitary rotation of one subspace cannot affect the overall performance. Despite of the fact that the orthogonality itself puts the HXC (in MAC phase) on the same level as the classical MAC with joint decoding of both data streams, the HDF strategy with such HXC can still utilize all the BC phase benefits of network-coding principles (see e.g., [6] for details), regardless of the MAC phase channel parametrization.

Example design process for generally *nonorthogonal* E-PHXC codebooks $\mathcal{B}_A, \mathcal{B}_B$ is presented in Algorithm 1. Some examples of nonorthogonal binary codebooks, which were found using this algorithm, are presented in Table 3. The construction algorithm, however, does not guarantee zero-mean nor equal distance (Gram matrix) codebooks $\mathcal{B}_A, \mathcal{B}_B$. It is obvious that if the alphabet $\mathcal{B}_i$ satisfies the design criteria from Theorem 9, then the codebook $\mathcal{B}'_i = -\mathcal{B}_i$ (all codewords have inverted signs) satisfies the design criteria as well (this holds for any alphabet cardinality). The nonzero mean of any codebook can hence be quite easily adjusted by sequential swapping of the codebooks $\mathcal{B}_i$ and $-\mathcal{B}_i$ at the particular source, since the resulting "compound" codebook will be zero mean.

We have defined the E-PHXC codebooks $(\mathcal{B}_A, \mathcal{B}_B)$ in such a way that the shape of the HDFD decision regions is α-invariant. This was achieved by forcing all pairwise boundaries from $\mathcal{S}_{\mathrm{CB}}$ to be α-invariant. Note that only the *shape* of the HDFD decision regions was considered, hence it is possible that two hierarchical codewords $\mathbf{u}^k$, $\mathbf{u}^l$ switch their position in the constellation space (with respect to the corresponding pairwise boundary $\mathcal{R}^{kl}$) for some values of α. This phenomenon is affected only by the signs of real and imaginary part of α, so the relay HDF decoder must take into account at most four different patterns (one for each of the four possible sign combinations of $\Re\{\alpha\}$ and $\Im\{\alpha\}$) for hierarchical codewords. Note that the shape of the HDFD decision regions still remains α-invariant for arbitrary α, which is obvious from Figure 7, where the effect of the parameter sign is exemplified (for various values of α) on the example codebook I from Table 2.

5. Minimum Distance-Based Design Criteria for Higher-Order Cardinality Codebooks

The new challenge in the codebook design arises when we need to design a codebook with higher cardinality. It can be shown that the strictness of the complete E-PHXC design criteria ((21), (22), and (23)) disables the codebook design in $\mathbb{C}^2$ for higher than binary cardinality. To overcome this inconvenience, we will slightly "relax" the E-PHXC design criteria and propose a new codebook design algorithm which will provide the tool for the construction of codebooks with generally arbitrary cardinality. By relaxing the proposed design criteria; we lose the parameter-invariant shape of the decision regions at the relay HDF decoder, but nevertheless, the overall system performance does not have to be negatively influenced. As we will show in this section, the performance analysis of the codebooks constructed according to the modified design algorithm shows some promising performance (compared to the traditional linear modulation schemes—e.g., PSK, QAM).

5.1. Hierarchical Minimum Distance. As we have already mentioned in the introduction of this paper, the major problem of HDF strategy is the channel parametrization in the MAC phase of the bidirectional communication. Specific channel parametrization can invoke the eXclusive law [7]

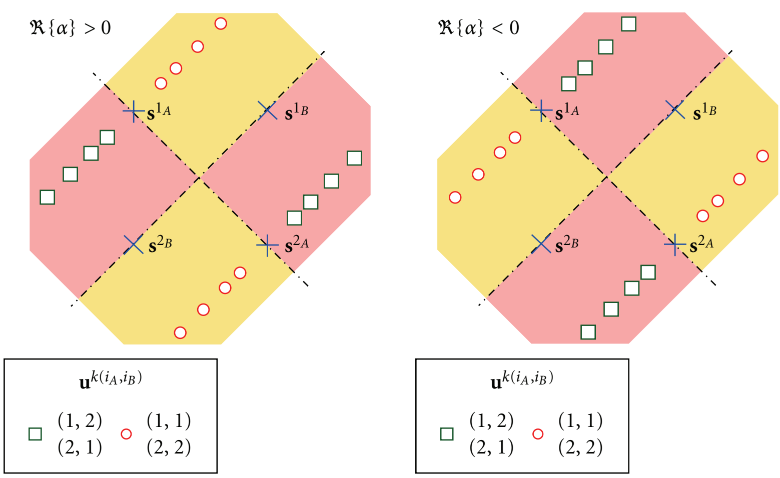


Figure 7: The sign of parameter α affects only the hierarchical codewords pattern, not the shape of the HDFD decision regions.

Table 2: Example binary E-PHXC codebooks.

	$\mathbf{s}^{1_A}$	$\mathbf{s}^{2_A}$	$\mathbf{s}^{1_B}$	$\mathbf{s}^{2_B}$
codebook I	$(-1, 1)$	$(1, -1)$	$(1, 1)$	$(-1, -1)$
codebook II	$(0, 1)$	$(0, -1)$	$(1, 0)$	$(-1, 0)$
codebook III	$(-1+j, 1-j)$	$(1-j, -1+j)$	$(1+j, 1+j)$	$(-1-j, -1-j)$

(1) Choose arbitrarily $\mathbf{s}^{1_B} \in \mathbb{C}^2$.
(2) Choose $\mathbf{s}^{2_B} \in \mathbb{C}^2$, $\mathbf{s}^{2_B} = \delta_1 \mathbf{s}^{1_B}$, where $\delta_1 \in \mathbb{C}^1$ is arbitrary scaling constant such that (25), (26) are satisfied.
(3) Find $\mathbf{v} \in \mathbb{C}^2$ such that $\langle \mathbf{v}; \mathbf{s}^{1_B} \rangle = 0$.
(4) Choose arbitrarily $\mathbf{s}^{1_A} \in \mathbb{C}^2$.
(5) Find $\mathbf{s}^{2_A} \in \mathbb{C}^2$ such that $\mathbf{s}^{2_A} = \mathbf{s}^{1_A} - \delta_2 \mathbf{v}$, where $\delta_2 \in \mathbb{C}^1$ is arbitrary scaling constant.
(6) $\mathcal{B}_A = \{\mathbf{s}^{1_A}, \mathbf{s}^{2_A}\}$, $\mathcal{B}_B = \{\mathbf{s}^{1_B}, \mathbf{s}^{2_B}\}$

Algorithm 1: Binary E-PHXC codebook—example design.

failures, resulting in significant performance degradation (see e.g., [7, 9]). This eXclusive law failures occur whenever the channel parametrization causes some pair of useful signals $(\mathbf{u}(\alpha), \mathbf{u}'(\alpha))$ which correspond to a distinct eXclusive relay output codeword $(\mathcal{C}(\mathbf{u}(\alpha)) \neq \mathcal{C}(\mathbf{u}'(\alpha)))$ to fall in (or close) to each other in the constellation space, thus increasing the probability of erroneous decision at the relay. These eXclusive law failures can be analyzed by observing the (squared) hierarchical distance of the useful signals in the constellation space

$$d^2_{(\mathbf{u},\mathbf{u}')}(\alpha) = \left\|\mathbf{u}(\alpha) - \mathbf{u}'(\alpha)\right\|^2. \tag{29}$$

For a general pair of useful signals $(\mathbf{u}^{(i_A,i_B)}, \mathbf{u}^{(i'_A,i'_B)})$, it becomes

$$d^2_{\mathbf{u}^{(i_A,i_B)},\mathbf{u}^{(i'_A,i'_B)}}(\alpha) = \left\|\left(\mathbf{s}^{i_A} - \mathbf{s}^{i'_A}\right) + \alpha\left(\mathbf{s}^{i_B} - \mathbf{s}^{i'_B}\right)\right\|^2. \tag{30}$$

The hierarchical minimum distance represents an approximation of the hierarchical decoder exact metric (as discussed, e.g., in [6]), and its performance is quite closely connected with the error rate performance of the whole system [6]. The hierarchical minimum distance for the HDF strategy can be defined as

$$d^2_{\min}(\alpha) = \min_{\mathcal{C}(\mathbf{u}) \neq \mathcal{C}(\mathbf{u}')} d^2_{(\mathbf{u},\mathbf{u}')}(\alpha). \tag{31}$$

The eXclusive law failures cause $d^2_{\min}(\alpha) \to 0$, which in turn results into a faulty decision of the relay decoder, and hence the performance degradation. In the following subsection we show that the fulfillment of (23) from the original E-PHXC design criteria is sufficient to avoid these failures for arbitrary channel parametrization.

5.2. Modified Design Criteria. Here we finally introduce the relaxed design criteria for the codebook construction. The following theorem shows that (23) is sufficient to avoid the significant performance degradation of the system by avoiding the eXclusive law failures ($d^2_{\min}(\alpha) = 0$).

Theorem 10. *The codebooks $\mathcal{B}_A = \{\mathbf{s}^{i_A}\}_{i_A}$, $\mathcal{B}_B = \{\mathbf{s}^{i_B}\}_{i_B}$ are resistant to the eXclusive law failures (for $|\alpha| > 0$) if the following condition holds:*

$$\left\langle \mathbf{s}^{i_A} - \mathbf{s}^{i'_A}; \mathbf{s}^{j_B} \right\rangle = 0 \quad \forall i_A < i'_A, \tag{32}$$

for all $i_A, i'_A, j_B \in \{1, 2, \ldots, N\}$.

Proof. It is obvious that (32) forces the following inner product to be always equal to zero:

$$\left\langle \left(\mathbf{s}^{i_A} - \mathbf{s}^{i'_A}\right); \left(\mathbf{s}^{j_B} - \mathbf{s}^{j'_B}\right) \right\rangle = 0. \tag{33}$$

Table 3: Example (nonorthogonal) binary E-PHXC codebooks.

	$\mathbf{s}^{1_A}$	$\mathbf{s}^{2_A}$	$\mathbf{s}^{1_B}$	$\mathbf{s}^{2_B}$
codebook IV	$(2, 1)$	$(1, 2)$	$(1, 1)$	$(-1, -1)$
codebook V	$(1, 2)$	$(1, 1)$	$(1, 0)$	$(-1, 0)$
codebook VI	$(1, j)$	$(j, 1)$	$(1 + j, 1 + j)$	$(-1 - j, -1 - j)$
codebook VII	$(2, 1 + j)$	$(1 + j, 2)$	$(1 + j, 1 + j)$	$(-1 - j, -1 - j)$
codebook VIII	$(2 + j, 1)$	$(1 + 2j, 2 - j)$	$(1 + j, 1 + j)$	$(-1 - j, -1 - j)$

(1) Choose $\mathbf{x}, \mathbf{y} \in \mathbb{C}^2$ such that $\langle \mathbf{x}; \mathbf{y} \rangle = 0$.
(2) $\mathcal{B}_B = \{q^{i_B} \cdot \mathbf{x}\}_{i_B=0}^{N-1}; \quad q^{i_B} \in \mathbb{C}$
(3) Pick $\mathbf{v} \in \mathbb{C}^2$.
(4) $\mathcal{B}_A = \{\mathbf{v} - q^{i_A} \cdot \mathbf{y}\}_{i_A=0}^{N-1}; \quad q^{i_A} \in \mathbb{C}$.

Algorithm 2: Higher-order codebook—example design.

Hence, the vectors $\Delta \mathbf{s}^{i_A, i'_A} = (\mathbf{s}^{i_A} - \mathbf{s}^{i'_A})$ and $\Delta \mathbf{s}^{j_B, j'_B} = (\mathbf{s}^{j_B} - \mathbf{s}^{j'_B})$ are mutually orthogonal. Now, since the pairs of mutually orthogonal vectors are always linearly independent (e.g., [15]), and the norm of the vector is equal to zero if and onlt if the vector is a zero vector ($\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{o}$), we can conclude that the minimum distance (30) will be nonzero for any $\alpha \neq 0$, because $(\mathbf{s}^{i_A} - \mathbf{s}^{i'_A}) + \alpha(\mathbf{s}^{i_B} - \mathbf{s}^{i'_B})$ is a linear combination of the linearly independent vectors. Hence, the eXclusive law failures $d^2_{\min}(\alpha) = 0$ are avoided for any $\alpha \neq 0$. □

The "relaxed" design criteria (32) are hence able to avoid the eXclusive law failures for any permissible value of the channel parametrization (excluding the singular case $\alpha = 0$). The Algorithm 2 presents an example design process for codebooks of generally arbitrary cardinality.

Vector $\mathbf{v}$ defines the mean of the codebook $\mathcal{B}_A$. For $\mathbf{v} = \mathbf{o}$, we obtain a trivial solution with mutually *orthogonal* codewords ($\langle \mathbf{s}^{i_A}; \mathbf{s}^{i_B} \rangle = 0$ for all $\mathbf{s}^{i_A}, \mathbf{s}^{i_B}$). In this case the main benefits of the HDF strategy are again mainly in the BC phase. For $\mathbf{v} \neq \mathbf{o}$, we have the codebook with a nonzero mean, which can be again easily adjusted by sequential swapping of the codebooks $\mathcal{B}_A$ and $-\mathcal{B}_A$. The coefficients q^{i_A}, q^{i_B} can be chosen from the classical linear modulation constellation (e.g., PSK or QAM) and can be generally identical ($q^{i_A} = q^{i_B}$) for both codebooks.

5.3. Performance Evaluation. Now we analyze the hierarchical minimum distance performance of the codebooks designed according to the Algorithm 2. Figures 8, 9, and 10 present the performance comparison of the example codebooks (with zero mean ($\mathbf{v} = \mathbf{o}$)) and classical linear modulation constellations (for various channel parametrization). All codebooks are scaled to have identical mean symbol energy. Note that the distance shortening at $|\alpha| \rightarrow 0$ is generally inevitable [6].

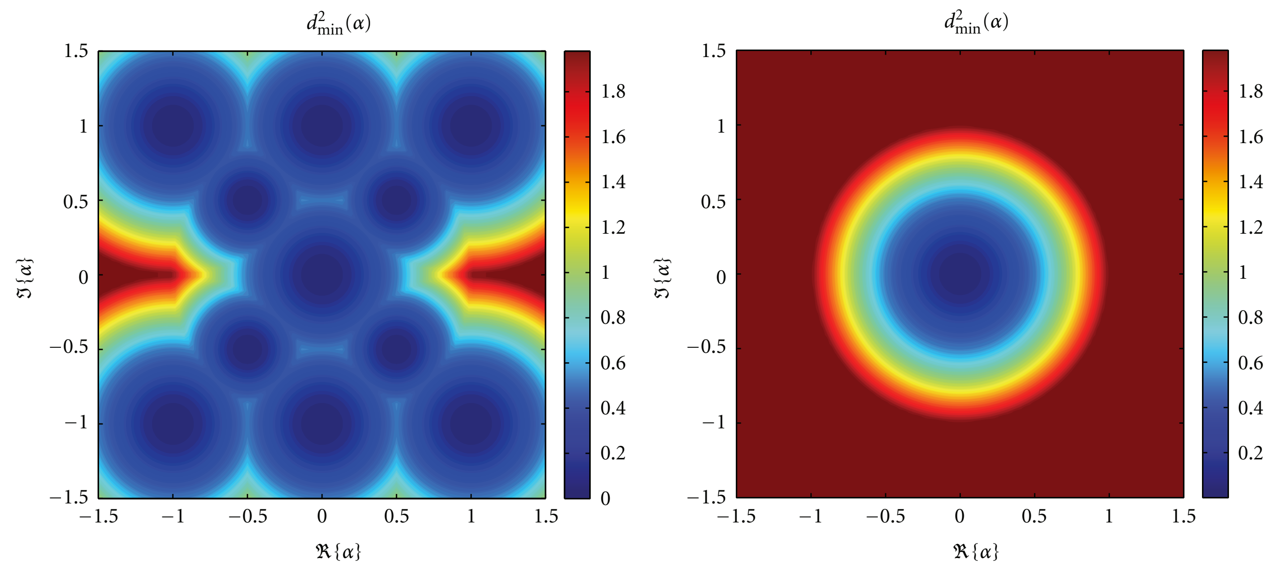
We conclude this section by observing the influence of the non-zero mean values of the codebook. In Figure 11, the comparison of the 4-ary example codebooks with $\|\mathbf{v}\| \in \{0, 1, 2\}$ is shown. It is obvious from this figure that the increasing value of the mean of the alphabet degrades the minimum distance performance.

6. Discussion of Results and Conclusion

The achievements of this paper can be summarized as follows. The MAC stage channel parametrization of the 2-WRC system with HDF strategy affects the decision regions at the relay as well as the overall system performance (which is influenced by the minimum distance performance of the chosen codebooks). The adverse effects of the channel parametrization (e.g., eXclusive law failures) can be generally avoided by the system adaptation (either by prerotation or by adaptive decision regions, see [6]), or by designing the source node codebooks in such a way that the decision regions at the relay are *invariant* to the channel parametrization. Since the adaptive solutions are generally not well suited for the fast-fading channels, we focus in this paper mainly on the second approach.

Utilizing the criterion for parameter-invariant constellation space boundary [5], we have derived E-PHXC *codebook construction criteria* that guarantee channel parameter-*invariant* relay decision regions. We have shown that these criteria require having two nonidentical source node codebooks. Strict nature of the full E-PHXC design criteria disables the possibility of designing the codebooks with higher than binary cardinality. To overcome this inconvenience, we have proposed the modified codebook construction algorithm (Algorithm 2), which is based on the relaxed version of the design criteria. This algorithm provides a feasible way for the design of codebooks with arbitrary cardinality.

Although neither of the construction algorithms require mutual orthogonality of the codebooks, it appears to be the simplest way of how to fulfill their requirements. Despite the fact that the orthogonality itself puts the HXC (in MAC phase) on the same level as the classical MAC with joint decoding of both data streams, the performance gain of the HDF strategy is in this case given by the increased reliability of the BC phase, which is available regardless of the MAC phase channel parametrization. Both proposed algorithms can produce a codebook with non-zero mean, which would have obvious performance disadvantages. It was shown that



(a)

(b)

Figure 8: Hierarchical minimum distance performance for QPSK and 4-ary example codebook (zero-mean).

this problem can be solved by sequential swapping of the codebooks $\mathcal{B}_i$ and $-\mathcal{B}_i$, since the resulting "compound" codebook will have zero-mean.

Performance analysis shows some promising results of the minimum distance of the example codebooks, compared to the classical linear modulation constellations. The more detailed investigation of the influence (pros and cons) of the modification of proposed E-PHXC design criteria on the relay processing/performance is a subject for future work.

Appendices

A. Proof of Lemma 4

We apply the PHXC design criteria (8), (9) to all critical boundaries. The critical boundary $\mathcal{R}^{kl}_{\mathrm{CB}}(\alpha)$ is the pairwise boundary between hierarchical codewords $\mathbf{u}^{k(i_A,i_B)}(\alpha)$ and $\mathbf{u}^{l(i'_A,i'_B)}(\alpha)$ where $i_A = i'_A$ or $i_B = i'_B$ (from Definition 2).

Now we have (from (8))

$$0 = \left\langle \mathbf{s}^{i_A} - \mathbf{s}^{i_A}; \mathbf{s}^{i_B} + \mathbf{s}^{i'_B} \right\rangle = \left\langle \mathbf{0}; \mathbf{s}^{i_B} + \mathbf{s}^{i'_B} \right\rangle, \tag{A.1}$$

for all $i_A = i'_A$, $i_B \neq i'_B$, $i_A, i_B, i'_B \in \{1, 2, \ldots, N\}$ and

$$0 = \left\langle \mathbf{s}^{i_A} - \mathbf{s}^{i'_A}; \mathbf{s}^{i_B} + \mathbf{s}^{i_B} \right\rangle = \left\langle \mathbf{s}^{i_A} - \mathbf{s}^{i'_A}; 2\mathbf{s}^{i_B} \right\rangle, \tag{A.2}$$

for all $i_B = i'{}_B$, $i_A \neq i'_A$, $i_A, i'_A, i_B \in \{1, 2, \ldots, N\}$.

From (9), we have

$$0 = \left\langle \mathbf{s}^{i_B} - \mathbf{s}^{i'_B}; \mathbf{s}^{i_B} + \mathbf{s}^{i'_B} \right\rangle, \tag{A.3}$$

for all $i_B \neq i'_B$, $i_B, i'_B \in \{1, 2, \ldots, N\}$ and

$$0 = \left\langle \mathbf{s}^{i_B} - \mathbf{s}^{i_B}; \mathbf{s}^{i_B} + \mathbf{s}^{i_B} \right\rangle = \left\langle \mathbf{0}; 2\mathbf{s}^{i_B} \right\rangle, \tag{A.4}$$

for all $i_B = i'_B$, $i_B, i'_B \in \{1, 2, \ldots, N\}$.

It is obvious that the inner products in (A.1) and (A.4) are always zero, and hence these conditions are always satisfied for all required individual codeword indices. From the remaining two inner products (A.2) and (A.3), we have the following criteria for the E-PHXC design:

$$\left\langle \mathbf{s}^{i_A} - \mathbf{s}^{i'_A}; \mathbf{s}^{i_B} \right\rangle = 0 \quad \forall i_A, i'_A, i_B \in \{1, 2, \ldots, N\},\ i_A \neq i'_A, \tag{A.5}$$

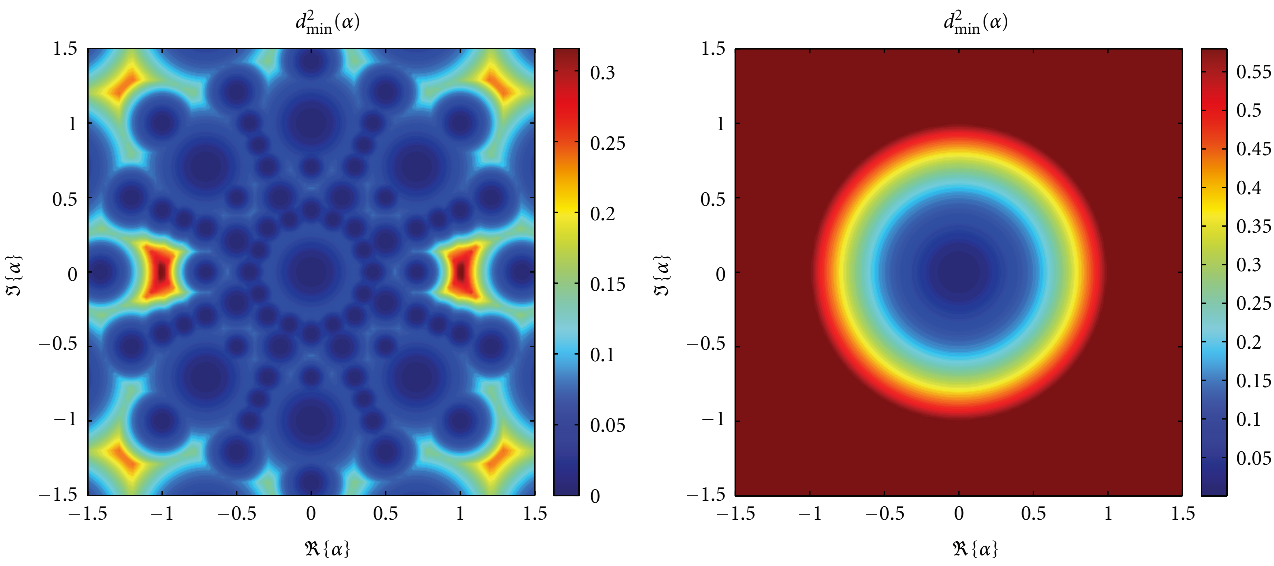
$$\left\langle \mathbf{s}^{i_B} - \mathbf{s}^{i'_B}; \mathbf{s}^{i_B} + \mathbf{s}^{i'_B} \right\rangle = 0 \quad \forall i_B, i'_B \in \{1, 2, \ldots, N\},\ i_B \neq i'_B. \tag{A.6}$$

Furthermore, the condition (A.5) for a given pair of indices (i_A, i'_A) is equivalent to the same condition for a "reversed" pair of these indices (i'_A, i_A), because $\langle \mathbf{s}^{i'_A} - \mathbf{s}^{i_A}; \mathbf{s}^{i_B} \rangle = -1 \langle \mathbf{s}^{i_A} - \mathbf{s}^{i'_A}; \mathbf{s}^{i_B} \rangle$ (and similarly for (A.6)). Hence it is sufficient to check (A.5) only for $i_A < i'_A$ (and (A.6) for $i_B < i'_B$).

B. Proof of Lemma 5

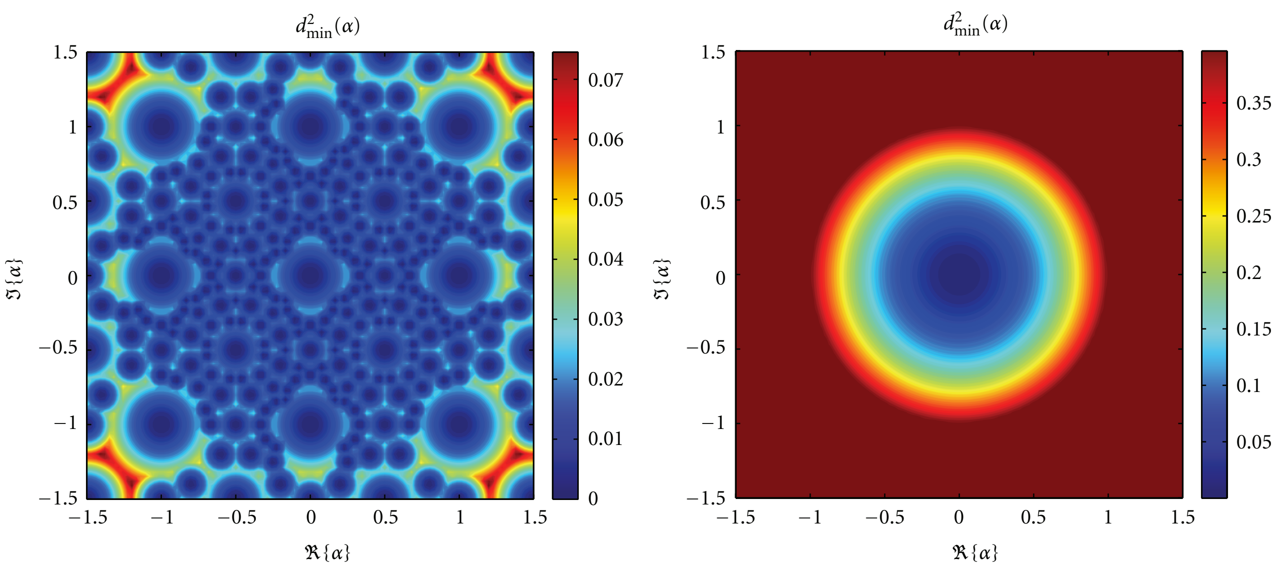
We choose (without loss of generality) two hierarchical codewords ($\mathbf{u}^{(i_{A1},i_{B1})}$ and $\mathbf{u}^{(i_{A2},i_{B2})}$) which have different indices ($i_{A1} \neq i_{A2}$ and $i_{B1} \neq i_{B2}$). These hierarchical codewords reside in a different row and column of the hierarchical codeword table (Table 1). The corresponding pairwise boundary is not considered critical by Definition 2 ($\mathcal{R}^{(i_{A1},i_{B1}),(i_{A2},i_{B2})} \notin \mathcal{S}_{\mathrm{CB}}$), hence it is not directly forced to be parameter-invariant by E-PHXC design criteria (see Figure 12). We will prove that $\mathcal{R}^{(i_{A1},i_{B1}),(i_{A2},i_{B2})}$ will be parameter-invariant if the E-PHXC design criteria are satisfied.

Assume that we have E-PHXC codebooks $\mathcal{B}_A$, $\mathcal{B}_B$. Then any hierarchical codeword pair residing in the same row or column of the corresponding hierarchical codeword table has



(a) (b)

FIGURE 9: Hierarchical minimum distance performance for 8-PSK and 8-ary example codebook (zero mean).



(a) (b)

FIGURE 10: Hierarchical minimum distance performance for 16-QAM and 16-ary example codebook (zero mean).

the pairwise boundary invariant to channel parameter (corresponding boundary is considered critical by Definition 2 and is required to be parameter-invariant by Definition 3). All such boundaries satisfy the PHXC pairwise criteria (8), (9). The following four boundaries are hence parameter-invariant (marked as $\mathcal{S}_{\mathrm{CB}}$ in Figure 12)

$$\begin{aligned} \mathcal{R}^{(i_{A1},i_{B1}),(i_{A2},i_{B1})} &= \mathcal{R}^{13}, \\ \mathcal{R}^{(i_{A1},i_{B1}),(i_{A1},i_{B2})} &= \mathcal{R}^{12}, \\ \mathcal{R}^{(i_{A1},i_{B2}),(i_{A2},i_{B2})} &= \mathcal{R}^{24}, \\ \mathcal{R}^{(i_{A2},i_{B1}),(i_{A2},i_{B2})} &= \mathcal{R}^{34}. \end{aligned} \tag{B.1}$$

Boundaries $\mathcal{R}^{13}, \mathcal{R}^{12}, \mathcal{R}^{24}$, and $\mathcal{R}^{34}$ satisfy the PHXC design criteria (8), (9), and hence the following three inner products (conditions for $\mathcal{R}^{12}$ and $\mathcal{R}^{34}$ are identical) are forced to be zero:

$$\left\langle \mathbf{s}^{i_{A1}} - \mathbf{s}^{i_{A2}}; \mathbf{s}^{i_{B1}} \right\rangle = 0, \tag{B.2}$$

$$\left\langle \mathbf{s}^{i_{B1}} - \mathbf{s}^{i_{B2}}; \mathbf{s}^{i_{B1}} + \mathbf{s}^{i_{B2}} \right\rangle = 0, \tag{B.3}$$

$$\left\langle \mathbf{s}^{i_{A1}} - \mathbf{s}^{i_{A2}}; \mathbf{s}^{i_{B2}} \right\rangle = 0. \tag{B.4}$$

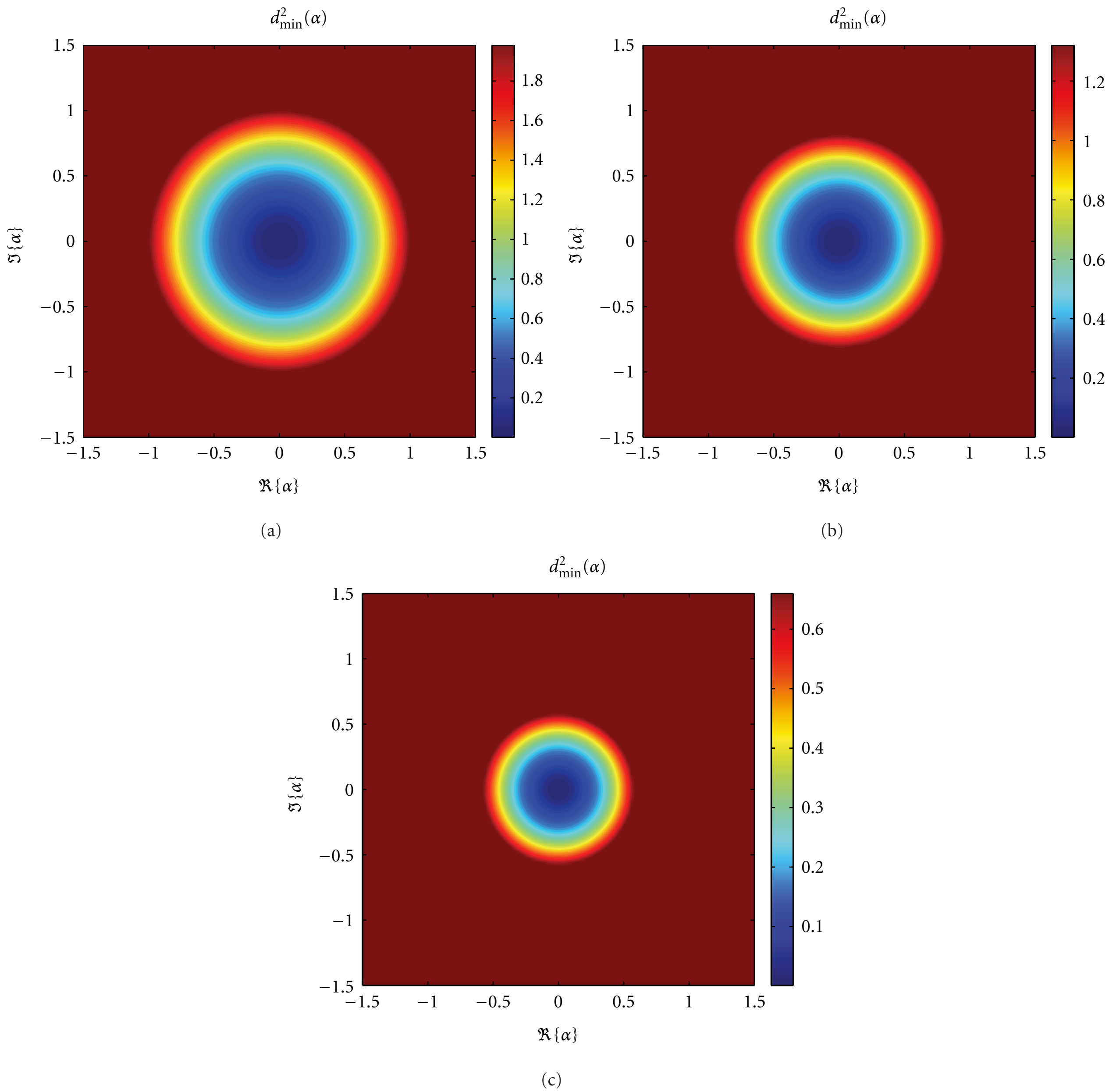


Figure 11: Hierarchical minimum distance for 4-ary example codebooks ($\|\mathbf{v}\| \in \{0, 1, 2\}$).

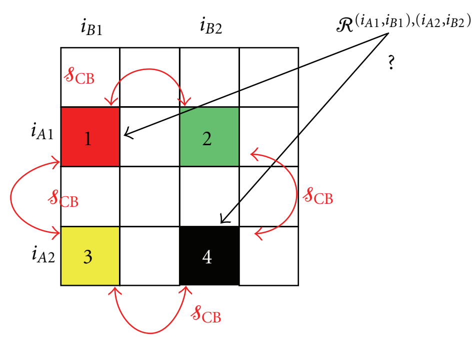


Figure 12: Impact of E-PHXC design criteria on noncritical ($\mathcal{R}^{k,l} \notin \mathcal{S}_{\mathrm{CB}}$) boundaries.

The examined pairwise boundary ($\mathcal{R}^{(i_{A1},i_{B1}),(i_{A2},i_{B2})} = \mathcal{R}^{14}$) will be parameter-invariant if the following two inner products are zero:

$$\left\langle \mathbf{s}^{i_{A1}} - \mathbf{s}^{i_{A2}}; \mathbf{s}^{i_{B1}} + \mathbf{s}^{i_{B2}} \right\rangle = 0, \tag{B.5}$$

$$\left\langle \mathbf{s}^{i_{B1}} - \mathbf{s}^{i_{B2}}; \mathbf{s}^{i_{B1}} + \mathbf{s}^{i_{B2}} \right\rangle = 0. \tag{B.6}$$

Now it is obvious that (B.6) is identical with (B.3) and (B.5) and is forced to be zero by (B.2) and (B.4):

$$\begin{aligned} \left\langle \mathbf{s}^{i_{A1}} - \mathbf{s}^{i_{A2}}; \mathbf{s}^{i_{B1}} + \mathbf{s}^{i_{B2}} \right\rangle &= \left\langle \mathbf{s}^{i_{A1}} - \mathbf{s}^{i_{A2}}; \mathbf{s}^{i_{B1}} \right\rangle + \left\langle \mathbf{s}^{i_{A1}} - \mathbf{s}^{i_{A2}}; \mathbf{s}^{i_{B2}} \right\rangle \\ \left\langle \mathbf{s}^{i_{A1}} - \mathbf{s}^{i_{A2}}; \mathbf{s}^{i_{B1}} + \mathbf{s}^{i_{B2}} \right\rangle &= 0. \end{aligned} \tag{B.7}$$

The pairwise boundary $\mathcal{R}^{(i_{A1},i_{B1}),(i_{A2},i_{B2})}$ satisfies both (B.5) and (B.6), and hence it is indirectly forced to be parameter-invariant by the E-PHXC design criteria (10), (11). In the same way we can prove that any permissible pairwise boundary ($\mathcal{R}^{k,l} \in \mathcal{S}_{\mathrm{PB}}$) with arbitrary indices k, l is forced to be parameter-invariant by the E-PHXC design criteria.

Acknowledgments

This paper was supported by the FP7-ICT SAPHYRE project, the Grant Agency of the Czech Republic, Grant no. 102/09/1624, and the Ministry of Education, Youth and Sports of the Czech Republic, program no. MSM6840770014, Grant OC188 and by the Grant Agency of the Czech Technical University in Prague, Grant no. SGS10/287/OHK3/3T/13.

References

[1] P. Larsson, N. Johansson, and K.-E. Sunell, "Coded bi-directional relaying," in *Proceedings of the IEEE 63rd Vehicular Technology Conference (VTC '06)*, pp. 851–855, Melbourne, Australia, 2006.

[2] S. J. Kim, P. Mitran, and V. Tarokh, "Performance bounds for bidirectional coded cooperation protocols," *IEEE Transactions on Information Theory*, vol. 54, no. 11, pp. 5235–5241, 2008.

[3] R. W. Yeung, S.-Y. R. Li, N. Cai, and Z. Zhang, *Network Coding Theory*, Now Publishers, Hanover, Mass, USA, 2006.

[4] P. Popovski and T. Koike-Akino, "Coded bidirectional relaying in wireless networks," in *Advances in Wireless Communications*, V. Tarokh, Ed., Springer, Berlin, Germany, 2009.

[5] J. Sykora, "Design criteria for parametric hierarchical exclusive constellation space code for wireless 2-way relay channel," in *Proceedings of the 8th COST 2100 Management Committee Meeting*, pp. 1–6, Valencia, Spain, May 2009, TD-09-855.

[6] T. Koike-Akino, P. Popovski, and V. Tarokh, "Optimized constellations for two-way wireless relaying with physical network coding," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 5, pp. 773–787, 2009.

[7] T. Koike-Akino, P. Popovski, and V. Tarokh, "Denoising maps and constellations for wireless network coding in two-way relaying systems," in *Proceedings of the IEEE Global Telecommunications Conference (GLOBECOM '08)*, pp. 3790–3794, New Orleans, La, USA, December 2008.

[8] J. Sykora and A. Burr, "Hierarchical exclusive codebook design using exclusive alphabet and its capacity regions for HDF strategy in parametric wireless 2-WRC," in *Proceedings of the 9th Management Committee Meeting*, pp. 1–9, Vienna, Austria, September 2009, TD-09-933.

[9] J. Sykora and A. Burr, "Hierarchical alphabet and parametric channel constrained capacity regions for HDF strategy in parametric wireless 2-WRC," in *Proceedings of the IEEE Wireless Communications and Networking Conference (WCNC '10)*, pp. 1–6, Sydney, Australia, April 2010.

[10] B. Rankov and A. Wittneben, "Achievable rate regions for the two-way relay channel," in *Proceedings of the IEEE International Symposium on Information Theory (ISIT '06)*, pp. 1668–1672, Seattle, Wash, USA, July 2006.

[11] I.-J. Baik and S.-Y. Chung, "Network coding for two-way relay channels using lattices," in *Proceedings of the IEEE International Conference on Communications (ICC '08)*, pp. 3898–3902, Beijing, China, May 2008.

[12] W. Nam, S.-Y. Chung, and Y. H. Lee, "Capacity bounds for two-way relay channels," in *Proceedings of the International Zurich Seminar on Communications (IZS '08)*, pp. 144–147, Zurich, Switzerland, March 2008.

[13] U. Erez and R. Zamir, "Achieving 1/2 log (1 + SNR) on the AWGN channel with lattice encoding and decoding," *IEEE Transactions on Information Theory*, vol. 50, no. 10, pp. 2293–2314, 2004.

[14] T. Koike-Akino, P. Popovski, and V. Tarokh, "Denoising strategy for convolutionally-coded bidirectional relaying," in *Proceedings of the IEEE International Conference on Communications (ICC '09)*, Dresden, Germany, June 2009.

[15] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, Pa, USA, 2001.