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On the Geometrical Characteristics of Three-Dimensional Wireless Ad Hoc Networks and Their Applications

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In a wireless ad hoc network, messages are transmitted, received, and forwarded in a finite geometrical region and the transmission of messages is highly dependent on the locations of the nodes. Therefore the study of geometrical relationship between nodes in wireless ad hoc networks is of fundamental importance in the network architecture design and performance evaluation. However, most previous works concentrated on the networks deployed in the two-dimensional region or in the infinite three-dimensional space, while in many cases wireless ad hoc networks are deployed in the finite three-dimensional space. In this paper, we analyze the geometrical characteristics of the three-dimensional wireless ad hoc network in a finite space in the framework of random graph and deduce an expression to calculate the distance probability distribution between network nodes that are independently and uniformly distributed in a finite cuboid space. Based on the theoretical result, we present some meaningful results on the finite three-dimensional network performance, including the node degree and the max-flow capacity. Furthermore, we investigate some approximation properties of the distance probability distribution function derived in the paper.

1. INTRODUCTION

A wireless ad hoc network can be considered as one consisting of a collection of nodes, and the relationship between them is peer to peer. That is to say, it adopts a non centralized and self-organized structure. On the one hand, in contrast to other networks, all the nodes in wireless ad hoc networks can transmit, receive, and forward messages, thus it does not require supports of the backbone networks. These characteristics make it superior to those schemes requiring infrastructure supports in respect of fast deploying at relatively low cost. Thereby, it may be especially useful in battlefield, disaster relief, scientific exploration, and so forth. On the other hand, the locations of nodes are random, which makes it more difficult to analyze the performance of wireless ad hoc networks.

Generally, wireless ad hoc networks can be modelled in the framework of random graph. Nodes and links of a network are considered as vertices and edges of a random graph $G(V, E)$, respectively, where V is the set of vertices, each with a random location, and E is the set of existing edges between vertices. In the symmetrical case, all nodes of the network are assumed to have the same transmission power and thus the same covering radius R, which is determined by the inverse power law model of attenuation and a predetermined threshold of power level for successful reception [1], that is,

$$P_0\left(\frac{R}{d_0}\right)^{-n} = P_{\text{threshold}}, \tag{1}$$

where P_0 is the power received at a close-in reference point in the far-field region of the antenna at a small distance d_0 from the transmitting antenna and n is the path loses component depending on the environment. In the model, there exists an edge (or a link) between node s and node t if the distance between them $d(s,t)$ is not larger than the covering radius R (Figure 1). Both Ramamoorthy et al. [2] and Li [3] adopted such kind of models, called random symmetric planar point graph and random geometric graph, respectively. Further studies go along in the framework of random graph theory. For example, Li [3] studied network connectivity and Ramamoorthy et al. [2] studied max-flow capacity analysis of network coding. The random graph model provides a meaningful framework for analyzing the wireless ad hoc network, especially when its topology is random.

It is obvious that the distance $d(s,t)$ between nodes s and t in the wireless ad hoc network is of great importance for further investigations. According to the model above, $d(s,t)$ determines whether two randomly chosen nodes s and t are

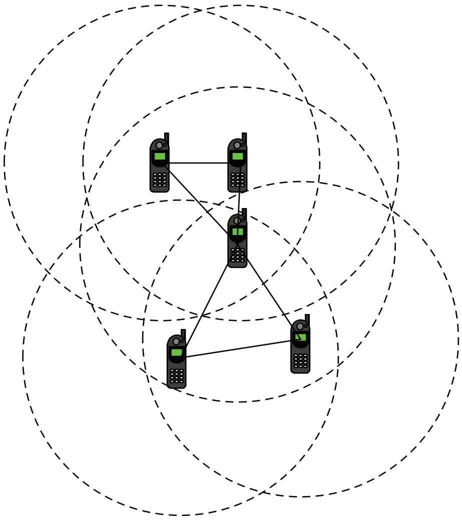


Figure 1: Random graph model of wireless ad hoc network.

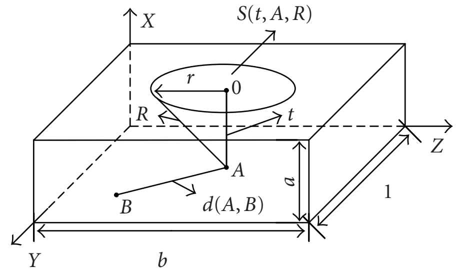


Figure 2: Illustration for the three-dimensional cuboid space.

able to communicate with each other directly with a given covering radius, and it also determines the characteristics of the whole network, such as the network topology and the max-flow capacity. Moreover, in their landmark paper, Gupta and Kumar [4] took into account the distance between the source and terminal of messages in measuring the transport capacity of wireless networks. Since the probability distribution gives a relatively thorough description of a random variable, in [5], we analyzed the distance probability distribution between nodes in the finite two-dimensional region under the assumption that they are independent and uniformly distributed, and presented the results for the rectangular region and the hexagonal region. In this paper, we will study the case of the finite three-dimensional wireless networks, which represents a wide category of practical networks, such as those deployed in the air space, in a building, or in other three-dimensional sensor networks. A formula to calculate the distance probability distribution between nodes in a finite cuboid space is deduced.

The node degree, defined as the number of a node's neighbors with which the node can communicate directly without relay, is an important measure of network. It describes local connectivity and also influences global properties. For networks in the infinite two-dimensional region, based on the inverse power law model of attenuation with lognormal shadowing fading, Orriss and Barton [6] proved that the number of audible stations of a station, corresponding to the node degree in this paper, obeys the Poisson distribution. This also comprehends the special case of random graph model above, which does not allow random shadowing. Verdone [7] extended the discussion to the infinite three-dimensional space and got the same conclusion. However, there are many differences between networks in the finite space and those in the infinite space due to the edge effect of the finite region [2]. For a wireless ad hoc network in the finite two-dimensional region, we presented, in [5], that the probability distribution of node degree is much more complicated, even in the absence of random shadowing. In this paper, we will extend the result in [5] to the finite three-dimensional network.

The max-flow capacity of a network [8] is another important performance measure and is the upper bound of transmission capacity of a network. In the single-source single-terminal transmission, Ahuja et al. [9] proved that the max-flow capacity between the source node and the terminal node can be achieved only by routing. And in the single-source multiple-terminal transmissions, Ahlswede et al. [10] and Li et al. [11] showed that the global max-flow capacity, which is the minimum of the max-flows between each pair of source and terminal, can be achieved by applying network coding. Ramamoorthy et al. [2] investigated the capacity of network coding for random networks by studying the relationship between max-flow capacity of network and the probability of links' existence in random networks in a unit square region. In this paper, based on the random graph model, we will present further results on the max-flow capacity of the three-dimensional networks in a finite space, under the assumption that each link has unit capacity.

The rest of this paper is organized as follows. In Section 2, we deduce the distance probability distribution function between nodes that are independent and uniformly distributed in cuboid space. Then, on the basis of the distribution function, some meaningful results on the wireless ad hoc network characteristics are presented, including the node degree in Section 3 and the max-flow capacity in Section 4. Some numerical results are presented in Section 5 on the approximation property of the distance probability distribution function. Finally, we conclude the paper in Section 6.

2. DISTANCE PROBABILITY DISTRIBUTION BETWEEN NODES IN CUBOID SPACE

As mentioned in Section 1, the study of the distance probability distribution is of great importance for further study. In this section, we discuss the probability distribution of distance between nodes in a cuboid space under the assumption that all nodes are independent and uniformly distributed in the space.

As shown in Figure 2, let A and B denote two arbitrary nodes in cuboid space C of $a \times 1 \times b$ ($a \le 1 \le b$). The distance between A and B is denoted by $d(A, B)$ and its probability distribution by $F(R) = P(d(A, B) \le R)$. In [5], we presented the probability distribution of distance between nodes in the rectangular region, which is the basis for the discussion of the three-dimensional case. The results in [5] is as follows.

Theorem 1. *Let A' and B' be two points which are independent and uniformly distributed in a $1 \times b$ $(1 \leq b)$ rectangular region. Then the probability distribution function of distance between A' and B', that is, $f(r) = P(d(A', B') \leq r)$, is*

$$f(r) = \frac{E_{A'}[S(A', r)]}{S}, \tag{2}$$

or equivalently

$$E_{A'}[S(A', r)] = S \times f(r), \tag{3}$$

where $S = 1 \times b$ denotes the area of the rectangular region, $S(A', r)$ denotes the area where Disc(A', r) *and the rectangular region overlap, where* Disc(A', r) *represents a disc with radius r and center A', and $E_{A'}[S(A', r)]$ denotes the expectation of $S(A', r)$ in location of A'. Further, the expression of $f(r)$ is as follows:*

$$f(r) = \begin{cases} f_1(r) = \dfrac{\pi r^2}{b} - \dfrac{4(b+1)r^3}{3b^2} + \dfrac{r^4}{2b^2}, \quad 0 \leq r < 1, \\ f_2(r) = \dfrac{b\pi - 1}{b^2} r^2 - \dfrac{2r^2}{b} \arccos\dfrac{1}{r} + \dfrac{2(2r^2+1)}{3b}\sqrt{r^2-1} - \dfrac{4}{3b} r^3 + \dfrac{1}{6b^2}, \quad 1 \leq r < b, \\ f_3(r) = \dfrac{\pi r^2}{b} - \dfrac{2r^2}{b}\left(\arccos\dfrac{b}{r} + \arccos\dfrac{1}{r}\right) + \dfrac{2(2r^2+1)}{3b}\sqrt{r^2-1} \\ \qquad + \dfrac{2(2r^2+b^2)}{3b^2}\sqrt{r^2-b^2} - \dfrac{r^4}{2b^2} - \dfrac{b^2+1}{b^2} r^2 + \dfrac{b^4+1}{6b^2}, \quad b \leq r < \sqrt{b^2+1}, \\ f_4(r) = 1, \quad r \geq \sqrt{b^2+1}. \end{cases} \tag{4}$$

Now come back to the three-dimensional case. First, let A be settled and B uniformly distributed in the cuboid space C, as shown in Figure 2. Let Sphere(A, R) denote a sphere with center A and radius R, and $V(A, R)$ denote the volume of the space where Sphere(A, R) and the cuboid space C overlap. It is obvious that the probability of $d(A, B) \leq R$, denoted by $F_A(R)$, is equal to that one where point B falls inside Sphere(A, R). Thus,

$$F_A(R) = \frac{V(A, R)}{V}, \tag{5}$$

where $V = a \times b$ denotes the volume of cuboid C. Furthermore, if point A is also uniformly distributed in the cuboid C and is independent of the location of point B, the probability of $d(A, B) \leq R$, denoted by $F(R)$, is the expectation of $F_A(R)$ with uniformly distributed location of point A, that is,

$$F(R) = E_A[F_A(R)] = E_A\left[\frac{V(A, R)}{V}\right]. \tag{6}$$

Next, we will discuss the calculation of $F(R)$. Let $S(t, A, R)$ denote the area of an X-axial cross section of the space where Sphere(A, R) and the cuboid space C overlap, with distance t from A. Thus,

$$V(A, R) = \int_T S(t, A, R)\,dt, \tag{7}$$

where T denotes the integral region of t. Then, we get the following expression:

$$\begin{aligned} F(R) &= E_A\left[\frac{V(A, R)}{V}\right] \\ &= E_A\left[\frac{1}{ab}\int_T S(t, A, R)\,dt\right] \\ &= \frac{1}{ab}\int_T E_A[S(t, A, R)]\,dt \\ &= \frac{1}{ab}\int_T P(t, A, R)[b \times f(r)]\,dt \\ &= \frac{1}{a}\int_T P(t, A, R) f(r)\,dt, \end{aligned} \tag{8}$$

where r denotes the radius of the X-axial cross section of Sphere(A, R) with distance t from the sphere center A, and $E_A[S(t, A, R)] = P(t, A, R)[b \times f(r)]$ denotes the expectation of $S(t, A, R)$ in random location of A. $P(t, A, R)$ is defined as the probability that the center of the X-axial cross section of Sphere(A, R) with distance t from A is inside the cuboidal space C, which reflects the distribution of point A along the X-axis. And $b \times f(r)$ denotes the expectation of $S(t, A, R)$ with the qualification that the center of the section is inside the cuboid space C, which is derived from (3) and reflects the distribution of A in the $Y - Z$ plane. Thus, the problem in three-dimensional space can be reduced to the combination of one in one-dimension and one in two-dimension. It is obvious that

$$r = \sqrt{R^2 - t^2}, \quad |t| \le R,$$
$$P(t, A, R) = \begin{cases} \dfrac{1}{a}(a - |t|), & |t| \le a; \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

Thus,

$$F(R) = \frac{1}{a}\int_T \frac{a - |t|}{a} f(r)dt = \frac{2}{a}\int_{T^+} \frac{a - t}{a} f(r)dt, \tag{10}$$

where T^+ denotes the integral region of $t \ge 0$. Further, substituting t with $\sqrt{R^2 - r^2}$, we have

$$F(R) = \frac{2}{a}\int_{E_r} P(r) f(r)dr, \quad P(r) = \frac{r}{\sqrt{R^2 - r^2}} - \frac{r}{a}, \tag{11}$$

where $f(r)$ is as presented in (4) and integral region E_r is $[0, R]$ for the case $R < a$ and $[\sqrt{R^2 - a^2}, R]$ for $R \ge a$, respectively. It is not hard to see that $F(R)$ has the following two different expressions:

$$\begin{aligned} F(R) &= \frac{2}{a}\int_{E_r} P(r) f(r)dr \\ &= \begin{cases} F_1(R), & a \le 1 \le b \text{ and } \sqrt{a^2+1} < b; \\ F_2(R), & a \le 1 \le b \text{ and } \sqrt{a^2+1} \ge b, \end{cases} \end{aligned} \tag{12}$$

where $F_1(R)$ and $F_2(R)$ each have segmented expressions, that is,

$$F_1(R) = \begin{cases} \dfrac{2}{a}\displaystyle\int_0^R P(r) f_1(r)dr, & 0 \le R < a, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^R P(r) f_1(r)dr, & a \le R < 1, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^1 P(r) f_1(r)dr + \dfrac{2}{a}\displaystyle\int_1^R P(r) f_2(r)dr, & 1 \le R < \sqrt{1+a^2}, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^R P(r) f_2(r)dr, & \sqrt{1+a^2} \le R < b, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^b P(r) f_2(r)dr + \dfrac{2}{a}\displaystyle\int_b^R P(r) f_3(r)dr, & b \le R < \sqrt{a^2+b^2}, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^R P(r) f_3(r)dr, & \sqrt{a^2+b^2} \le R < \sqrt{1+b^2}, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^{\sqrt{1+b^2}} P(r) f_3(r)dr + \dfrac{2}{a}\displaystyle\int_{\sqrt{1+b^2}}^R P(r) f_4(r)dr, & \sqrt{1+b^2} \le R < \sqrt{1+a^2+b^2}, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^R P(r) f_4(r)dr, & \sqrt{1+a^2+b^2} \le R, \end{cases}$$

$$F_2(R) = \begin{cases} \dfrac{2}{a}\displaystyle\int_0^R P(r) f_1(r)dr, & 0 \le R < a, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^R P(r) f_1(r)dr, & a \le R < 1, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^1 P(r) f_1(r)dr + \dfrac{2}{a}\displaystyle\int_1^R P(r) f_2(r)dr, & 1 \le R < b, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^1 P(r) f_1(r)dr + \dfrac{2}{a}\displaystyle\int_1^b P(r) f_2(r)dr & \\ \qquad + \dfrac{2}{a}\displaystyle\int_b^R P(r) f_3(r)dr, & b \le R < \sqrt{1+a^2}, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^b P(r) f_2(r)dr + \dfrac{2}{a}\displaystyle\int_b^R P(r) f_3(r)dr, & \sqrt{1+a^2} \le R < \sqrt{a^2+b^2}, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^R P(r) f_3(r)dr, & \sqrt{a^2+b^2} \le R < \sqrt{1+b^2}, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^{\sqrt{1+b^2}} P(r) f_3(r)dr + \dfrac{2}{a}\displaystyle\int_{\sqrt{1+b^2}}^R P(r) f_4(r)dr, & \sqrt{1+b^2} \le R < \sqrt{1+a^2+b^2}, \\ \dfrac{2}{a}\displaystyle\int_{\sqrt{R^2-a^2}}^R P(r) f_4(r)dr, & \sqrt{1+a^2+b^2} \le R. \end{cases} \tag{13}$$

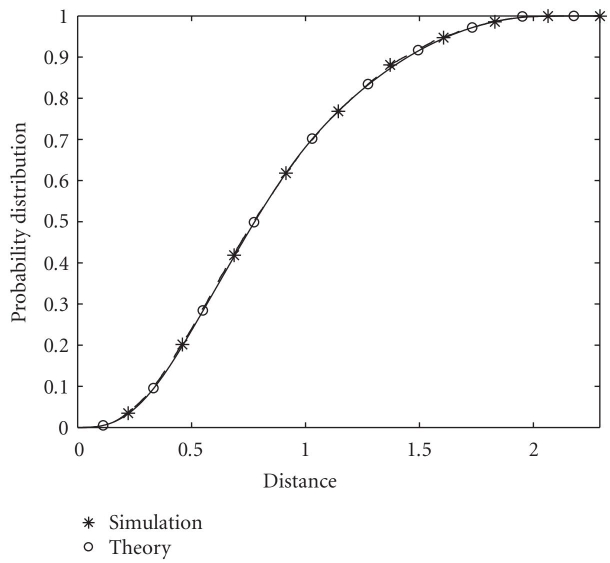


Figure 3: The distance probability distribution between nodes that are independent and uniformly distributed in a $0.5 \times 1 \times 2$ cuboid space. The curve marked by "∗" is the result of simulation and that by "∘" is the theoretical result.

Hitherto, we have given the expression to calculate the distance probability distribution between nodes that are independent and uniformly distributed in an $a \times 1 \times b$ cuboid space. It is not hard to derive an explicit formula from the above expressions. Moreover, since $F(R)$ is a single integral and both $P(r)$ and $f(r)$ have relatively simple expressions, one can use the above expressions to get his required results through numerical method in practice, instead of using the complicated explicit expression. Hence, we omit the explicit expression in detail here.

Simulation is conducted in a $0.5\times1\times2$ cuboid space. Each time, two nodes are generated independently and uniformly in the space and the distance between them are calculated. A total of 10000 such trials are carried out. The simulation and theoretical results on the distance probability distributions between nodes are plotted in Figure 3, which demonstrates the correctness of our theoretical expression.

3. NODE DEGREE

Recall that in a wireless ad hoc network, the degree of a node is defined as the number of its neighbors, that is, the number of nodes that can receive its message directly without relay [3]. It is obvious that one node's degree is equivalent to the number of the nodes located in its power covering range except itself (Figure 1). From the viewpoint of successful exchange of messages, node degree is an important factor which represents the local topological status of the wireless ad hoc network. To a certain extent, the node locations and their corresponding degrees would affect the configuration of the wireless ad hoc network and even the total network throughput. Verdone [7] proved that in an infinite three-dimensional space, the node degree obeys the Poisson distribution. However, in the case of a finite space, the explicit expression of the probability distribution of the node degree is more complicated, even in the absence of random shadowing. In this section, we will discuss this problem based on the result in Section 2.

Suppose the nodes of an n-node wireless ad hoc network are independent and uniformly distributed in a cuboid space with the same covering radius R. According to the discussion in Section 2, for an arbitrary settled node $A = (x, y, z)$ in the network, it is obvious that its degree obeys the binomial distribution with parameters $n-1$ and $F_A(R)$, that is,

$$P_A\{d(A)=k\} = \binom{n-1}{k} F_A(R)^k \left[1-F_A(R)\right]^{(n-k-1)},$$
$$F_A(R) = \frac{V(A,R)}{V}, \tag{14}$$

where $V(A,R)$, V, and $F_A(R)$ are as defined in Section 2. Furthermore, if node A is uniformly distributed in the finite space, the probability distribution of node degree can be formulated as follows:

$$\begin{aligned} P\{d(A)=k\} &= E_A\left[P_A\{d(A)=k\}\right] \\ &= E_A\left[\binom{n-1}{k} F_A(r)^k \left[1-F_A(R)\right]^{(n-k-1)}\right]. \end{aligned} \tag{15}$$

Based on the expression, one can calculate the probability distribution of the node degree through numerical method. In the terminology of communication, such a probability distribution equals the probability distribution of the number of nodes with which a randomly chosen node can communicate directly. In the symmetrical case, where all nodes have the same covering radius, this probability distribution also equals that of the number of nodes that may interfere with the reception of a certain nodes if they transmit signals simultaneously.

Note that $P\{d(A)=k\}$ is neither the binomial distribution with parameter $F(R)$, that is,

$$P\{d(A)=k\} = \binom{n-1}{k} F(R)^k \left[1-F(R)\right]^{(n-k-1)}, \tag{16}$$

nor the widely used Poisson distribution with parameter $\lambda = (n-1)F(R)$, that is,

$$P\{d(A)=k\} = \frac{\lambda^k}{k!} e^{-\lambda}, \tag{17}$$

where $F(R)$ is the distance probability distribution between nodes, which is equal to the probability that two randomly chosen nodes can communicate with each other directly

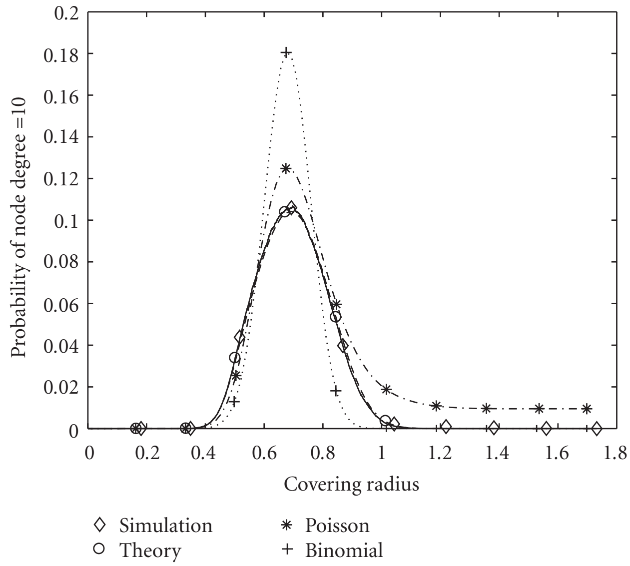


FIGURE 4: The probability of the node degree equals 10 in a 20-node network in $1 \times 1 \times 1$ cuboid space.

within the covering radius R. Though the Binomial distribution and the Poisson distribution seem reasonable at first sight, both of them would lead to significant bias in the case of the finite three-dimensional space, while our theoretical result in (15) agrees with that of the simulation, as shown in Figure 4. It is worth noticing that the expectation of $d(A)$ has the same expression as that of the binomial distribution, which is given as follows:

$$\begin{aligned} E[d(A)] &= \sum_{k=0}^{n-1} kE_A[P_A\{d(A)=k\}] \\ &= E_A\left[\sum_{k=0}^{n-1} kP_A\{d(A)=k\}\right] \\ &= E_A[(n-1)F_A(R)] \\ &= (n-1)F(R). \end{aligned} \tag{18}$$

Simulation about node degree are carried out in a $1\times1\times1$ rectangular region and 100 nodes are deployed each time. The results of the mean values of the node degrees derived from simulation and theory are shown in Figure 5. It indicates that our theoretical results have a good matching with that of the simulation.

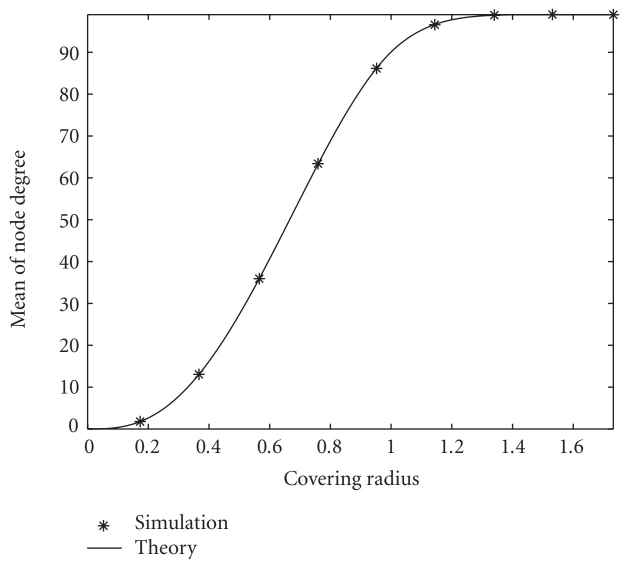
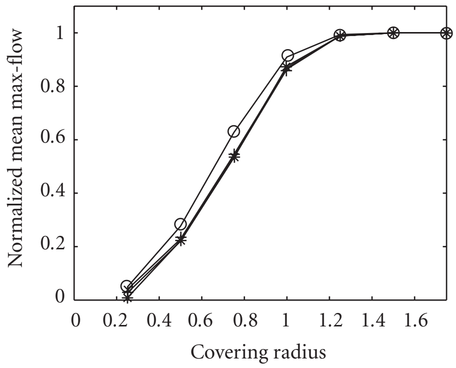


FIGURE 5: The average value of node degree of 100-node network in $1 \times 1 \times 1$ cuboid space.

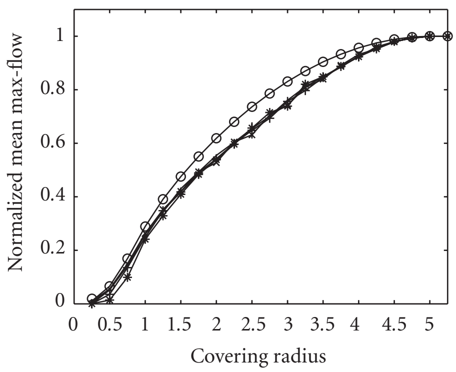
4. NETWORK CAPACITY

As mentioned above, the max-flow capacity is another important parameter on the performance of network. However, in the case of wireless ad hoc networks, it is extremely difficult to formulate the max-flow capacity as a simple expression due to the dependence among the wireless links as mentioned in [2]. In this section, we discuss this problem and present two results, partly based on large amount of simulations.

Figure 6 presents some examples of simulations and illustrates our observations. The simulations are set in the cuboid spaces of various sizes, and various node densities are checked for each size combination. It is assumed that all nodes in a network have the same covering radius R, and that each of all existing links has unit capacity. Max-flows are computed for some source-destination pairs that are



(a)



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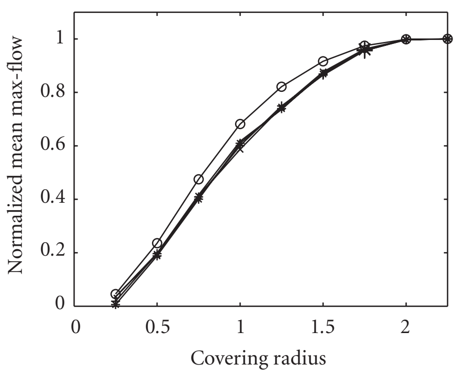

(c)

FIGURE 6: The normalized max-flow capacity of networks in the cuboid spaces, (a) $1 \times 1 \times 1$, (b) $1 \times 1 \times 5$, and (c) $0.5 \times 1 \times 2$.

randomly chosen in each random graph, and the algorithm of the C++ program follows that in [12]. The mean value of the max-flows is then normalized by $(n-1)$, where n is the number of nodes in the network. There are two meaningful results, which are the extensions of those in [5].

Firstly, the mean max-flow capacity of an n-node network is approximately $(n-1)k(R)$, where $k(R)$ is a function in covering radius R. This can be illustrated by the fact that the normalized mean values of the max-flows for the networks with different node numbers but the same covering radius are approximately the same, especially as the node number (or rather the node density in space) and the covering radius are relatively large. In other words, there exists a linear relationship between the mean value of the max-flow and the node number of a wireless ad hoc network when the covering radius keeps constant. Another important observation is that, if the covering radius R is relatively small, the normalized mean value of the max-flow decreases sharply once the node number (or node density) falls below a threshold, which results in a poor connection of the network. This can be explained intuitively: a network tends to collapse into some independent components, with no connection existence between any two of them, which makes the number of disconnected node pairs increase sharply and hence the mean value of the max-flow capacity decreases. The quantified and more exact depiction of the phenomenon requires further investigation and is beyond this paper.

Secondly, $k(R)$ is no greater than the probability distribution of distance between nodes, that is, $F(R)$, as illustrated in Figure 6. This conclusion can be proved by using some basic results in graph theory as follows. It is well known that the max-flow of a graph is equal to its min-cut, and that for a network with unit link capacity, its min-cut is no greater than the minimal degree of source node and terminal node as in [8]. As shown in Section 3, $F(R)$ reflects the mean value of node degree, including that of the source and that of the terminal. Thus, it is easy to understand that $k(R)$ is no greater than $F(R)$. In fact, for any two positive random variables x and y, $\text{mean}[\min(x, y)] \leq \min[\text{mean}(x), \text{mean}(y)]$ is always true.

5. APPROXIMATION OF $F(R)$

Consider the case of the cuboid space where the length a is relatively small. It is easy to see that the three-dimensional case would reduce to the two-dimensional case as the value of a approaches to 0. Thus, it makes sense to use $f(r)$ for the two-dimensional case to approximate $F(R)$ for the three-dimensional case, as long as a is small enough. Our simulation suggests that the performance would be fairly nice when a/b is relatively small, as shown in Figure 7, while the bias of the approximation would become larger as a/b increases, as shown in Figure 8. The exact performance of this kind of approximation will be examined in this section in terms of both the relative error and the absolute error. Some simulation results are presented.

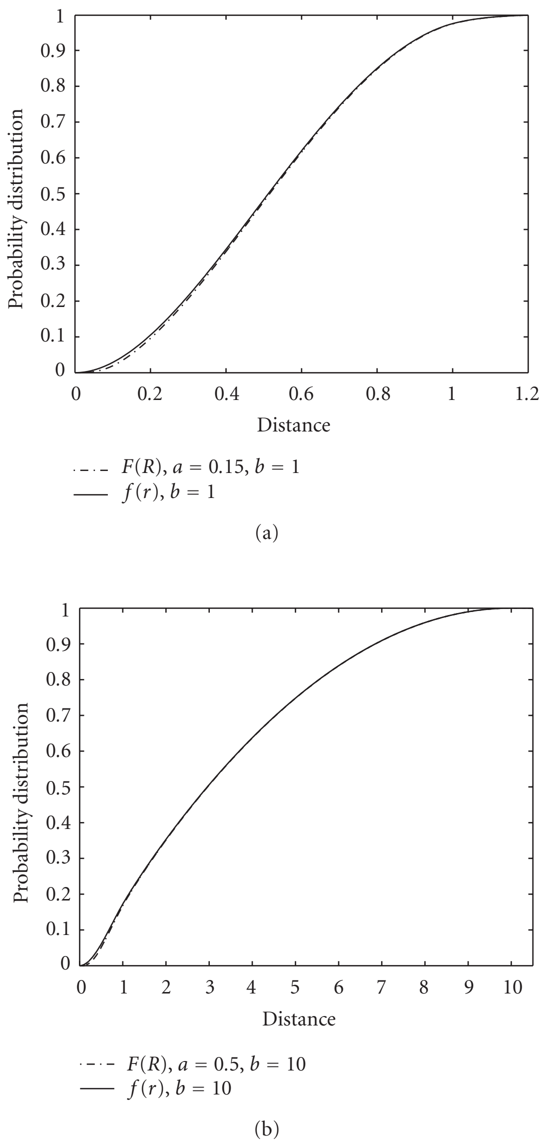

FIGURE 7: The comparison between $f(r)$ and $F(R)$ when $a = 0.15$, $b = 1$ (a) and $a = 0.5$, $b = 10$ (b). The label "Distance" refers to R for $F(R)$ and r for $f(r)$.

5.1. Relative error of the approximation

The relative error is defined as follows:

$$\text{Error}_{\text{Relative}} = \frac{|F(R) - f(r)|}{F(R)}\bigg|_{r=R}, \tag{19}$$

where $F(R)$ refers to the formula for the three-dimensional case and $f(r)$ refers to that for the two-dimensional case (Figure 9). We get the following observations from the simulation results.

(i) The relative error of approximation decreases as R increases.

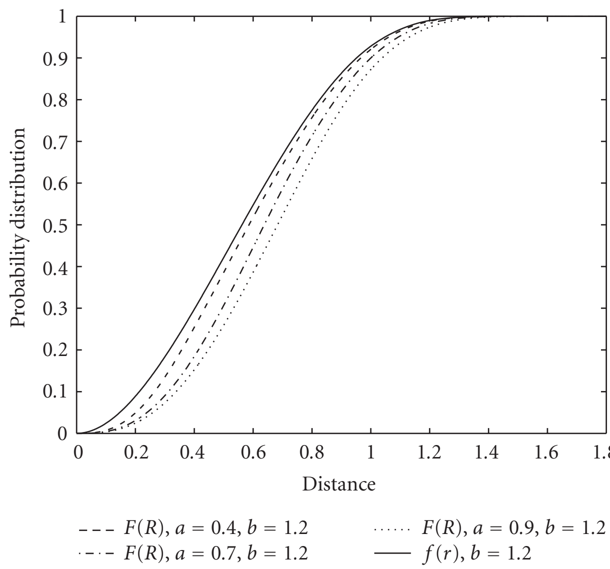

FIGURE 8: The comparison between $f(r)$ and $F(R)$ when a/b increases. The label "Distance" refers to R for $F(R)$ and r for $f(r)$.

(ii) The relative error of approximation, at a given relative distance R/a, increases as a/b decreases.

(iii) In the range $R > 1.8a$, the relative error is upper bounded by 5.5%.

(iv) In the range $R < 1.8a$, the relative error is becoming larger sharply, as R approaches 0.

5.2. Absolute error of the approximation

The absolute error is defined as follows:

$$\text{Error}_{\text{Absolute}} = |F(R) - f(r)|_{r=R}. \tag{20}$$

Simulation is carried out to examine the influence of a and b on the absolute error. Results are plotted in Figure 10, and we get the following observations.

(i) The absolute error increases as a increases.

(ii) The absolute error decreases as b increases.

(iii) The absolute error is upper bounded by 0.01 when $a < 0.15$.

(iv) The absolute error peaks occur at points being around $R = 0.86a$ for all cases.

6. CONCLUSIONS

In this paper, we investigated the probability distribution of distance between nodes independently and uniformly distributed in a finite three-dimensional cuboid space, and presented an explicit formula. Some meaningful observations about the wireless ad hoc network in the cuboid space were obtained, including the node degree and the max-flow

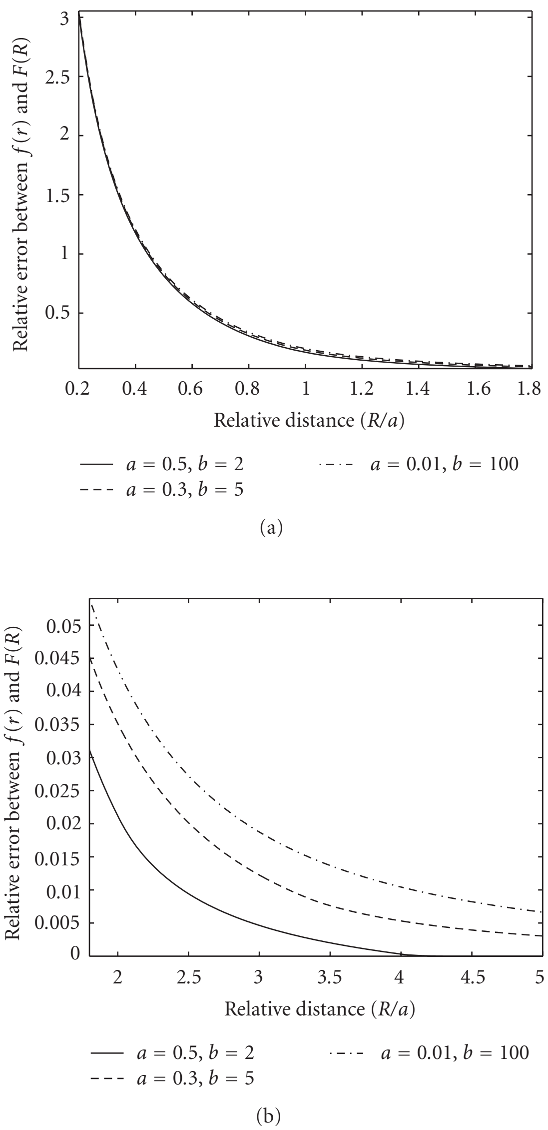

Figure 9: The relative error of using $f(r)$ to estimate $F(R)$ (a) for $0.1 < R/a < 1.8$ and (b) $1.8 < R/a < 5$.

capacity. The reduced dimensional approximation of the distance probability distribution between nodes are also presented.

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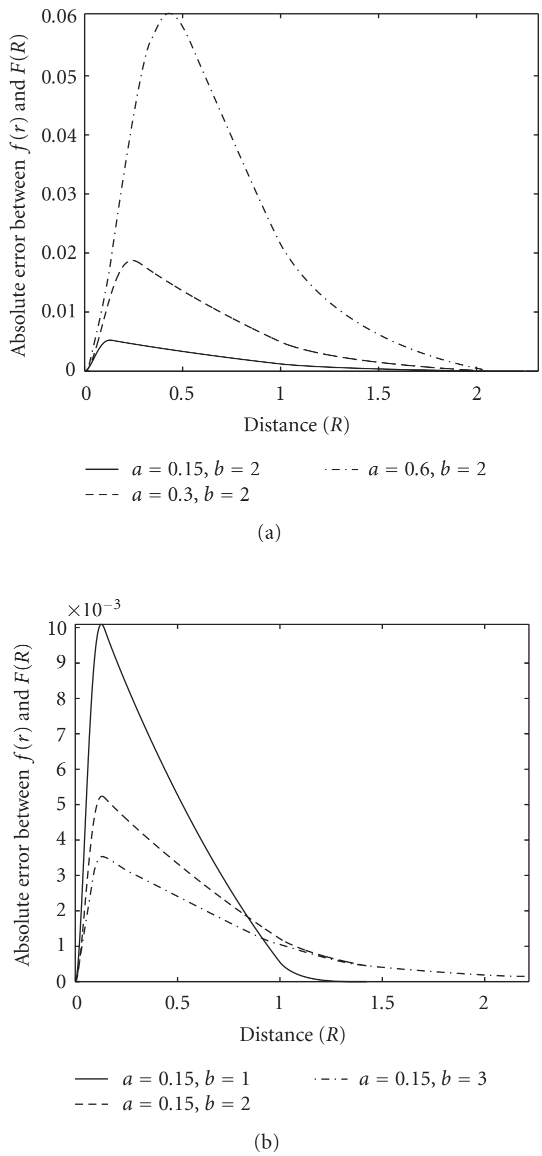

Figure 10: The absolute error of using $f(r)$ to estimate $F(R)$. The curves in (a) depict the absolute error with different value a, and the curves in (b) depict that with different value b.

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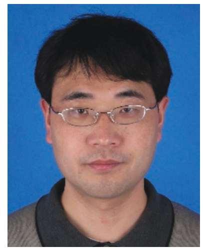

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