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Research Article

Spatial Multiplexing Gains for Realistic Sized Ad Hoc Networks with Directional Antenna Arrays

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We concentrate on an ad hoc network model with nodes on integer lattice points over a 2D plane. We examine the limits of ad hoc network performance for systems with antenna arrays capable of allowing both spatial multiplexing and directional processing. Two cases are considered. In the first case, we consider "perfect" directional antenna arrays, in other words, each node can form beams of infinitesimally narrow beamwidth. In this case, the throughput capacity of an ad hoc network is independent of the network size. In the second case, we consider a more practical system where each node can form a fixed number of beams of finite beamwidth. Our results show that the spatial multiplexing gains depend on the system size, antenna beamwidth, and number of antenna beams. Furthermore, we show that spatial multiplexing gains offsetting the interference-related performance degradation can be achieved in ad hoc networks with thousands of nodes.

1. INTRODUCTION

The application of multiple antennas at both the transmitter and receiver sides of a wireless system for the purpose of spatial multiplexing (simply put, spatial multiplexing in this context means making use of multiple paths distinct in physical space to deliver information from a source to the corresponding destination) [1, 2] has been shown to have the potential of achieving extraordinary bit rates. As a result, this topic has received significant study recently [3–10]. The issues of MAC/routing protocol design for ad hoc networks utilizing multiple antennas were also studied in [11–17]. It should be noted that antenna arrays can implement directional processing and beamforming in addition to spatial multiplexing. When these approaches are suitably combined, good network performance is achieved. However, the majority of research has focused on point-to-point communications. Here we study spatial multiplexing at the network level. Further, we assume the antenna arrays used for the spatial multiplexing will also be used for beamforming.

We will study the uniform throughput capacity, or simply uniform throughput, which we define as the minimum long-term average rate at which every node in the network can transmit to its corresponding destination. The throughput in wireless ad hoc networks is inherently limited by interference since the nodes have to use the common wireless channel in order to transmit different information. The use of multiple directed beams and spatial multiplexing cannot completely eliminate the interference but, as we will see, can greatly alleviate it, even if a small number of beams at every node is used.

Previous results [18] have shown poor performance for large ad hoc networks without spatial multiplexing. In this paper, we will show that spatial multiplexing provides large gains in throughput for small networks, and that while these gains shrink for larger networks, there are still spatial multiplexing gains in networks with thousands of nodes.

In this paper, we consider a network consisting of n nodes located on a square grid with periodic boundary conditions. We begin by examining a simpler case of infinitely narrow beamwidth where every beam is just a zero-width ray with the origin at the transmitting node. We find that in this case, the uniform throughput is upper bounded by $Wg/2$, where W is the rate of point-to-point transmission along a single beam, and g is the number of beams each transmitter can form. Furthermore, we show that, under reasonable assumptions, the uniform throughput of $Wg/2$ can be achieved regardless of the network size (and the distance between sources and destinations). Next, we consider the case of a finite angular beamwidth D where the beams are infinite

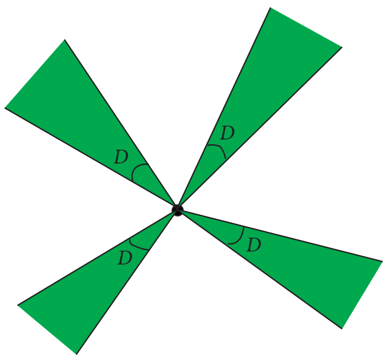

Figure 1: A node with 4 transmitting beams of angular width D.

two-dimensional cones with vertices at the transmitter. We show that in this case the uniform throughput is bounded from above by a quantity, that is proportional to Wg, and for larger network sizes, proportional to $\bar{l}'/\sqrt{n}$, where $\bar{l}'$ is the average of the longest g hops possible from a given node without interference. Moreover, a fixed fraction of this upper bound can be shown to be achievable. The result is that, although the degradation of performance due to interference is still present for the finite beamwidth case, the spatial multiplexing allows one to "postpone" the throughput from falling below W (which is what the throughput would be for just a single source-destination pair) until fairly large network sizes (thousands of nodes for beamwidth of about 10 degrees and no more than 10 beams) which makes practically large network sizes entirely feasible.

The directional antenna assumptions used in this paper are consistent with accepted results [19] that imply that antenna arrays (smart antennas) can be used to form beams in n different directions if at least n antennas are available in an array. Further, by proper spacing of the antennas and by employing more antennas, these beams can be made more narrow. Therefore, the number of antennas limits the number of directional beams that each node may employ.

The rest of the paper is organized as follows. In Section 2, we formulate the model used in the paper. Section 3 is devoted to evaluating the uniform throughput for both cases of infinitely narrow beamwidth and finite beamwidth. Section 4 contains conclusions.

2. SYSTEM DESCRIPTION

In this paper, we evaluate the uniform throughput among $n = m^2$ nodes with each node located at a unique integer point of the lattice $\Omega(m) = \{(a,b) \,|\, a,b = 1,2,\ldots,m\}$ covering an m-by-m square region with periodic boundary conditions (a torus).[1] We measure all the distances below in L_1 metric ("Manhattan distance") in units of lattice space, unless noted otherwise.

We assume that each transceiver node is equipped with an antenna array that can produce g antenna beams, each with angular width D (see Figure 1) such that $gD \le 2\pi$. Node-to-node transmissions on the torus are allowed only in the "shorter" direction, that is, the largest horizontal and vertical transmitting distance allowed by the model is $\lfloor m/2 \rfloor$. The latter requirement is used to imitate a real system with boundaries while disregarding boundary effects where they can lead to unwanted complications.

A transmission from node i to node j along a beam b_l^i, $l = 1,2,\ldots,g$, is assumed to be successful if[2]

(1) node j lies inside the beam b_l^i,

(2) node j does not lie inside any other beam $b_{l'}^k$ closer to the node k than the intended receiver.

If a transmission along a given beam is successful, the corresponding transmission rate is assumed to be equal to W. We assume that if a node-to-node transmission is successful, exactly one packet is transmitted.[3] We use the full-duplex assumption: a node can both transmit and receive up to g packets simultaneously from different directions (along different beams). Thus the maximum number of packets that a node can simultaneously handle[4] (either transmit or receive) is equal to $2g$.

Finally, we make the following assumption about the relative position of sources and destinations.

Assumption 1. We assume each source and destination are separated by a distance $m/2$ lattice spaces in the horizontal and vertical directions (and thus are separated by a distance of m in L_1 metric).

This assumption is used to simplify calculations, and by removing the "randomness" from them, make it possible to obtain quantitative results for networks of finite size as opposed to asymptotic results valid only in the limit $n \to \infty$.

In the following, we measure the throughput in units of W. Thus in order to obtain the throughput in conventional units of bits/s, all the following results should be multiplied by W.

3. THROUGHPUT WITH SPATIAL MULTIPLEXING

In this section, we explore the uniform throughput for the model described above.

[1] That is, as the coordinates are to be understood "module m." More precisely, the nodes with coordinates $(x+m, y)$ are identified with nodes with coordinates (x, y) for $y = 1,2,\ldots,m+1$ and nodes with coordinates $(x, m+y)$ are identified with nodes with coordinates (x, y) for $x = 1,2,\ldots,m+1$.

[2] This reception success model can be justified by assuming that nodes exercise power control so that the received powers are all the same.

[3] Thus all packets are assumed to be of size $W\delta t$, where δt is the time slot length.

[4] Note that, this is just an assumption made for the sake of convenience. For a real system that cannot simultaneously transmit and receive due to interference, the schedule described later in the paper can easily be modified so that each node's transmissions and receptions are separated in time (done in different time slots) and the overall capacity will simply pick up a factor of 1/2 compared to the results in this paper.

3.1. General bounds on throughput of ad hoc networks with spatial multiplexing

The uniform throughput of any network can be upper bounded on the basis of just the number of successful transmissions per time slot and the average number of node-to-node hops necessary to complete a source-to-destination transmission. Assume that any successful node-to-node transmission happens at a rate of W. Let s be the expected number of distinct successful transmissions per node in a time slot for the given transmission scheme. Also, let h_{ia} be the number of hops it takes to completely reach the destination from the source node i when using the path[5] a. Define $h_i = \min_a h_{ia}$ to be the length of the shortest path between node i and its destination. Finally, let $\overline{h} = (1/n)\sum_{i=1}^{n} h_i$ be the mean hop count of the shortest source-to-destination path taken over all nodes in the network. Then a simple upper bound on the uniform throughput can be obtained.

Theorem 1. *For any transmission scheme, the uniform throughput per node, $\mathcal{T}$, satisfies the inequality*

$$\mathcal{T} \le \frac{s}{\overline{h}}. \quad (1)$$

Proof. Let us consider a large number T of time slots. Then the total number of successful node-to-node transmissions over these T time slots is

$$N_T = snT. \quad (2)$$

On the other hand, in order to obtain a throughput of at least $\mathcal{T}$ for the source node i, one would need at least $\mathcal{T} h_i T$ successful node-to-node transmissions. Therefore, in order to obtain a throughput of at least $\mathcal{T}$ for all n sources, the corresponding count of successful node-to-node transmissions has to be at least

$$N_T^{\mathcal{T}} = \mathcal{T} T \sum_{i=1}^{n} h_i = n\mathcal{T}\overline{h}T. \quad (3)$$

It is clear then that we must have $N_T^{\mathcal{T}} \le N_T$, which implies, using (2) and (3), that

$$\mathcal{T} \le \frac{s}{\overline{h}}. \quad (4)$$

□

In the following, we will be interested in transmission schemes which allow for all nodes to successfully transmit using all available beams in every time slot,[6] that is, schemes for which $s = g$. Consider a given node i. Let V_i be the set of nodes such that if $j \in V_i$, then g nonoverlapping beams b_l^i, $l = 1, 2, \ldots, g$, originating at node i can be arranged in such a way that

(i) $j \in b_l^i$ for some value of l;
(ii) if $k \in b_l^i$ for $k \ne j$, then $r_{ij} < r_{ik}$ (where r_{ij} is the Euclidean distance between nodes i and j).

We will call nodes in V_i *visible without interference*, or *vwi*, to node i.

Let L_i be the distance between the source i and its destination. Also let $l_{i1}, l_{i2}, \ldots, l_{i|V_i|}$ be the distances from node i to nodes in the set V_i ordered so that $l_{i1} \ge l_{i2} \ge \cdots l_{i|V_i|}$. Let $\overline{l}_i' = (1/g)\sum_{j=1}^{g} l_{ij}$ be the mean of the g largest distances l_{ij}, and let $\overline{l}' = (1/n)\sum_{i=1}^{n} \overline{l}_i'$ be the mean of the quantities $\overline{l}_i'$. Then one can show that the uniform throughput $\mathcal{T}$ can be upper bounded as follows. (Here $\overline{L} = \sum_{i=1}^{n} L_i$.)

Theorem 2. *The uniform throughput $\mathcal{T}$ satisfies the inequality*

$$\mathcal{T} \le \frac{g\overline{l}'}{L}. \quad (5)$$

Proof. Consider a long time period (measured in time slots) T. During this time, in order to have a throughput of $\mathcal{T}$ for all sources, the total *information transport* (i.e., information transmitted over distance, measured in bit · m) of

$$C_T = n\mathcal{T}\overline{L}T \quad (6)$$

would be needed.

On the other hand, let us compute the largest information transport that can be achieved during the same time T. If in every time slot every node uses all its beams for successful transmission (thus transmitting to vwi nodes only), the largest information transport would be

$$\widehat{C}_T = \left(\sum_{i=1}^{n}\sum_{j=1}^{g} l_{ij}\right) T = ng\overline{l}'\,T. \quad (7)$$

Since $C_T \le \widehat{C}_T$, we see from (6) and (7) that

$$\mathcal{T} \le \frac{g\overline{l}'}{\overline{L}}. \quad (8)$$

□

Let us now explore the achievability of these upper bounds. Let Assumption 1 hold. In addition, let us assume that there exists a transmission scheme A such that

(i) every node in the network successfully transmits r packets every time slot;
(ii) each path from a source to the corresponding destination is at most $h_{\max}$ hops long;
(iii) the paths from each source to destination are the same (relative to the source and the corresponding destination) for every source node;
(iv) every node uses the same directions (hops) for its transmissions in every time slot.

Then we can make the following claim.

Theorem 3. *The uniform throughput achieved by the transmission scheme A satisfies the inequality*

$$\mathcal{T} \ge \frac{r}{h_{\max}}. \quad (9)$$

[5] We need the path index since for multiple antenna systems it may be possible to simultaneously transmit information to the same destination using different node-to-node transmission paths.
[6] It is easy to show, using geometric arguments, that such schemes yield the highest possible throughput.

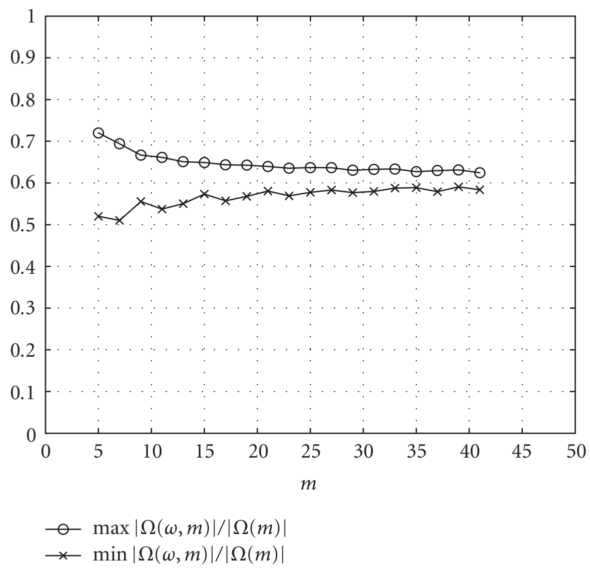

FIGURE 2: Numerical evaluation of $|\Omega(\omega, m)|/|\Omega(m)|$.

Proof. Due to symmetry between nodes (i.e., ensured by Assumption 1), the total number of source-destination paths passing through every node is the same. The total number of such paths is equal to rn, and the total number of links in them is at most $rnh_{\max}$. So, the number of paths passing through every node is at most $rh_{\max}$. Since every node can send and receive r packets in every time slot, there exists a schedule in which a node serves every path passing through it at least once in $h_{\max}$ time slots. This means that every source node can send its own packets at least once in $h_{\max}$ time slots using all r beams. By the definition of throughput, this implies that every source node can have a throughput of at least $r/h_{\max}$, which proves the theorem. □

3.2. *Infinitely narrow beamwidth*

Let us first consider the case of infinitely narrow beamwidth, that is, $D = 0$. In this case, the beams are just straight lines. Given m and an arbitrary point $\omega \in \Omega(m)$, let $\Omega(\omega, m)$ be the set of lattice points in $\Omega(m)$ that are vwi to ω. If the beamwidth is infinitely small, a node is vwi to another node as long as no other nodes lie on the line segment between them. It is shown in [20] that, for a square (with boundaries) lattice, regardless of ω,

$$\alpha = \lim_{m\to\infty} \frac{|\Omega(\omega, m)|}{|\Omega(m)|} = \frac{6}{\pi^2} \approx 0.6079. \tag{10}$$

This value can be thought of as the asymptotic fraction of nodes that are vwi to an arbitrary node on the grid. In the case of a torus, the number of nodes (lattice sites) that are vwi to a given node is obviously the same regardless of the lattice site ω. In fact, it is easy to see that for the network on an $m \times m$ torus, $|V_i| = \max_{\omega} |\Omega(\omega, m)|$, and therefore,

$$\lim_{m\to\infty} \frac{|V_i|}{m^2} = \frac{6}{\pi^2}. \tag{11}$$

Figure 2 shows the minimum and maximum values for the quantity $\Omega(\omega, m)/\Omega(m)$ for the square grid (with boundaries). We see that for a torus the ratio $|V_i|/m^2$ always stays above the limiting ratio $6/\pi^2$.

The following proposition shows that any node on the torus can communicate with any other node[7] in at most two hops.

Proposition 1. *Any node i can communicate with any other node j in two hops so that node-to-node transmissions are between vwi nodes.*

Proof. In order to prove the proposition, we only need to show that the intersection of sets V_i and V_j is nonempty for any pair of nodes i and j.

Using the set-theoretic equality

$$|V_i \cup V_j| = |V_i| + |V_j| - |V_i \cap V_j|, \tag{12}$$

we can write

$$|V_i \cap V_j| = |V_i| + |V_j| - |V_i \cup V_j|. \tag{13}$$

Since $|V_i \cup V_j| \le n$ and $|V_i| = |V_j| \ge \alpha n$, we conclude that

$$|V_i \cap V_j| \ge (2\alpha - 1)n > 0, \tag{14}$$

which proves the lemma. □

We can now state the upper bound on the throughput of a lattice ad hoc network with infinitely narrow beamwidth.

Theorem 4. *The uniform throughput $\mathcal{T}$ for an ad hoc network on a square lattice with g beams of zero width for each node satisfies the inequality*

$$\mathcal{T} \le \frac{g}{2}. \tag{15}$$

Proof. We use Theorem 1. In that theorem, $s \le g$, and, in order to take advantage of the spatial multiplexing, one needs at least 2 hops. So $\overline{h} \ge 2$, which proves the theorem. □

As to the achievability of the upper bound, it could in principle depend on the location of sources and destinations. If, as we assumed, they all are separated by a distance of $m/2$ in both vertical and horizontal directions, then a transmission strategy employing 2 hops for all g paths from every source to destination can be used. Then, as is easy to see, a throughput of exactly $g/2$ for all source-destination pairs can be achieved whenever $|V_i \cap V_{d(i)}| \ge g$ (where $d(i)$ stands for the destination of the source i). In other words, the necessary and sufficient condition for the achievability of throughput $g/2$ is the existence of at least g nodes that are vwi to both the source and the destination. Any g of these nodes can be used as relays. Thus we have the following theorem.

[7] In any real network, a tradeoff between the hop length and the error rate (and therefore throughput) would be present. Within the approximation adopted here we neglect these issues understanding that they would have to be considered in order to obtain practically applicable results.

Theorem 5. *For the lattice ad hoc network with the source-destination locations described in Assumption 1, the uniform throughput of* $g/2$ *is achievable provided* $|V_i \cap V_{d(i)}| \geq g$.

On the other hand, from Lemma 1, we know that $|V_i \cap V_j| \geq 2(\alpha - 1)n$ for any pair of nodes i and j. It follows that the conditions of Theorem 5 are satisfied as long as $(2\alpha - 1)n \geq g$. Noting also that, since the source-to-destination relative locations are the same for all source-destination pairs, all relay nodes will get the same number of packets to forward, we have the following corollary.

Corollary 1. *The uniform throughput of* $g/2$ *is achievable provided* $(2\alpha - 1)n \geq g$.

3.3. Finite beamwidth

Now, let the beamwidth be $D > 0$. As we will see, in this case the number of nodes that are vwi to a given node will not grow with the network size. Instead, it will be dependent on the beamwidth, resulting in the need for multiple hops in order to reach the destination.

The following theorem establishes a connection between the beamwidth D and the maximum distance $\widehat{H}$ to a vwi node.

Theorem 6. *The maximum beamwidth* $\widehat{D}$ *that can be used to transmit without interference to a node a distance H away is*

$$\widehat{D} = \arctan\left(\frac{1}{H-2}\right) \tag{16}$$

and the maximum distance for a transmission without interference given a beamwidth D is

$$\widehat{H} = 2 + \left\lfloor \frac{1}{\tan D} \right\rfloor, \tag{17}$$

for any $H \geq 3$ *and* $D \leq 90^\circ$.

Proof. Equation (16) can easily be seen to be correct for $H \leq 4$ by a straightforward enumeration of possibilities. In the following, we consider the case $H \geq 5$.

First, we show that $\widehat{D} \leq \arctan(1/H-2)$ for any vwi node at a distance H from the source. Let the source i be at the origin. Due to lattice symmetry, it is sufficient to only consider vwi nodes with coordinates $(H-\nu,\nu)$ where $\nu \leq (H-1/2)$ (the node with coordinates $(H/2, H/2)$ cannot be vwi unles $H = 2$). We consider cases $\nu \leq H/3$ and $\nu > H/3$ separately. □

Case $1 \leq \nu \leq H/3$

For the node $(H-\nu,\nu)$ to be vwi, the corresponding beam has to "clear" the nodes with coordinates $(H-\nu-1,\nu)$ and $(H-\nu-2,\nu-1)$. Let $\vec{u_a}$ and $\vec{u_b}$ be vectors with these coordinates, respectively (see Figure 3). Let us denote the angle between these vectors by θ_1. Also let $\theta_0 = \arctan(1/H-2)$. Thus we need to show that $\theta_1 \leq \theta_0$. Using the standard vector algebra, we see that

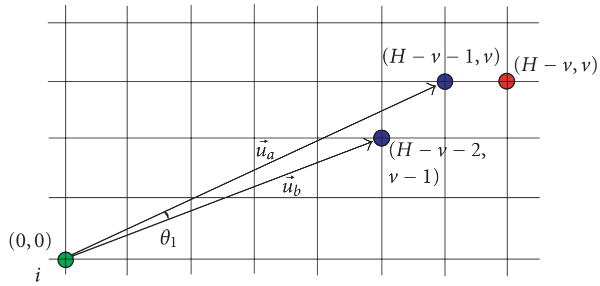

Figure 3: In order for the node $(H-\nu,\nu)$ to be vwi to the source at the origin, the corresponding beam has to "clear" nodes $(H-\nu-1,\nu)$ and $(H-\nu-2,\nu-1)$. In this figure, $H = 10$ and $\nu = 3$ so that $\nu < H/3$.

$$\cos^2(\theta_1) = \frac{\left((H-\nu-1)(H-\nu-2)+\nu(\nu-1)\right)^2}{\left((H-\nu-1)^2+\nu^2\right)\left((H-\nu-2)^2+(\nu-1)^2\right)}. \tag{18}$$

On the other hand, for the angle θ_0 we have

$$\cos^2(\theta_0) = \frac{(H-2)^2}{(H-2)^2+1}. \tag{19}$$

Subtracting, we obtain

$$\cos^2(\theta_0) - \cos^2(\theta_1) = \frac{d_1(\nu)}{\left((H-2)^2+1\right)\left((H-\nu-1)^2+\nu^2\right)\left((H-\nu-2)^2+(\nu-1)^2\right)}, \tag{20}$$

where

$$d_1(\nu) = -4\nu(\nu-1)\left(\left(\nu-\left(H-\frac{3}{2}\right)\right)^2 - \frac{1}{4}\right). \tag{21}$$

We see that $d_1(\nu) \leq 0$ for $\nu \leq H-2$ and $\nu \geq 1$. This implies that $\cos^2(\theta_0) - \cos^2(\theta_1) \leq 0$ in this range of ν. We conclude that $\theta_1 \leq \theta_0$ for all values of ν not exceeding $H/3$.

Case $H/3 < \nu \leq (H-1/2)$

In this case for the node $(H-\nu,\nu)$ to be vwi to the source at the origin, the corresponding beam has to "clear" the nodes $(H-\nu-2,\nu-1)$ and $(H-\nu-1,\nu-1)$. Again, let $\vec{u_b}$ and $\vec{u_c}$ be vectors with these respective coordinates (see Figure 4). Let θ_2 be the angle between these vectors. The use of standard vector algebra yields

$$\cos^2(\theta_2) = \frac{\left((H-\nu-1)(H-\nu-2)+(\nu-1)^2\right)^2}{\left((H-\nu-1)^2+(\nu-1)^2\right)\left((H-\nu-2)^2+(\nu-1)^2\right)}. \tag{22}$$

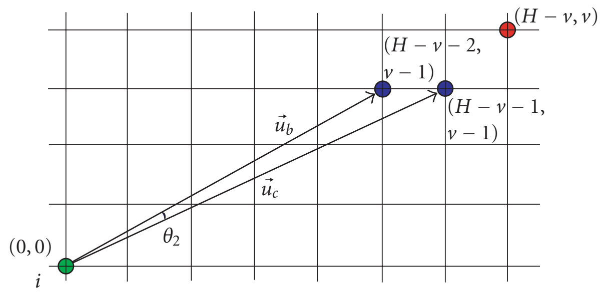

FIGURE 4: In order for the node $(H-v,v)$ to be vwi to the source at the origin, the corresponding beam has to "clear" nodes $(H-v-2,v-1)$ and $(H-v-1,v-1)$. In this figure, $H=11$ and $v=4$ so that $v>H/3$.

Subtracting $\cos^2(\theta_2)$ from $\cos^2(\theta_0)$, we obtain

$$\cos^2(\theta_0)-\cos^2(\theta_2) = \frac{d_2(v)}{((H-2)^2+1)((H-v-1)^2+(v-1)^2)((H-v-2)^2+(v-1)^2)}, \quad (23)$$

where

$$d_2(v) = -(H^2-4H+5)(H^2-2Hv-3H+2v^2+v+3)^2 + (H-2)^2((v-1)^2+(H-v-1)^2)((v-1)^2+(H-v-2)^2). \quad (24)$$

The second derivative of $d_2(v)$ is

$$d_2''(v) = -48\left(\left(v-\frac{H-1/2}{2}\right)^2+\frac{H^2-6H+15/2}{24}\right). \quad (25)$$

It is easy to see that $d_2''(v)<0$ everywhere as long as $H\geq 5$. This implies that the first derivative $d_2'(v)$ is monotonously decreasing everywhere and has one real root v_0. Setting $v=H-1/2$ and evaluating the first derivative, we obtain $d_2'(H-1/2)=H^3-6H^2+11H-6>0$, for $H>3$. This implies that $v_0>H-1/2$ and, therefore, $d_2'(v)>0$, for $v\leq H-1/2$. Hence, $d_2(v)$ is an increasing function for the whole interval $H/3<v\leq H-1/2$. On the other hand, by setting $v=H-1/2$ in the expression for $d_2(v)$ we obtain $d_2(H-1/2)=0$. Therefore, we conclude that $d_2(v)\leq 0$ for $H/3<v\leq H-1/2$ which means that, on this interval, $\theta_2\leq\theta_0$, and the inequality $\widehat{D}\leq\arctan(1/H-2)$ is valid for $H\geq 5$.

Moreover, if we consider the node $(H-v-1,1)$, we can easily see that this node is vwi for $D=\arctan(1/H-2)$, meaning that the bound $\widehat{D}\leq\arctan(1/H-2)$ is tight.

Finally, it follows directly from (16) that if the beamwidth D is given, then the maximum hop length $\widehat{H}$ to a vwi node can be found as

$$\widehat{H} = 2+\left\lfloor\frac{1}{\tan D}\right\rfloor. \quad (26)$$

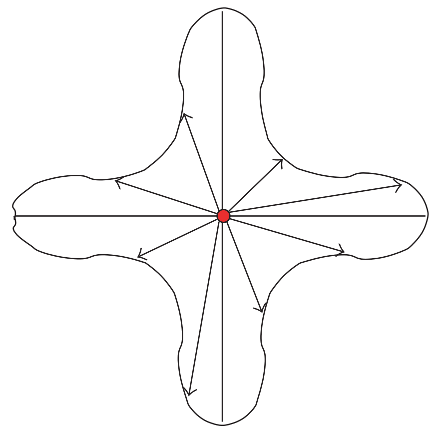

FIGURE 5: Envelope of nodes vwi to a source for finite beamwidth. Only the nodes inside the envelope can be vwi to the node in the origin.

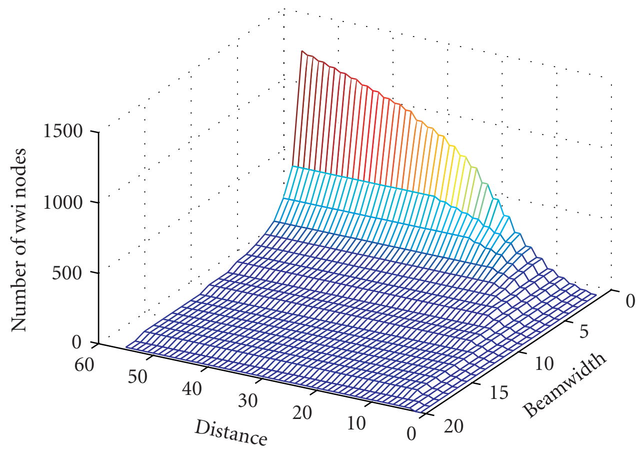

FIGURE 6: The dependence of the number of vwi nodes on the distance from the source and the beamwidth.

Theorem 6 described the largest L_1 distance for which a node is vwi to a source for a given beamwidth. It was also found that this maximum distance is achieved by a node whose position is one lattice point above the horizontal ($v=1$). Numerical evaluation shows that for larger beamwidths, almost all the nodes within a certain distance can be vwi. As the beamwidth is decreased, the nodes along the horizontal and vertical directions can be vwi disproportionately more. Therefore, for larger beamwidths, the envelope of vwi nodes looks like a diamond, and as the beamwidth decreases, the envelope becomes more and more cross-like in appearance (see Figure 5). Some nodes within this envelope cannot be vwi since some nodes may directly block other nodes. For example, a node at lattice point $(1,1)$ cannot "see" a node at lattice point $(3,3)$ since a node at $(2,2)$ is blocking it. We found the total number of nodes vwi to a source node for various distances and for various beamwidths (Figure 6), using numerical evaluation.

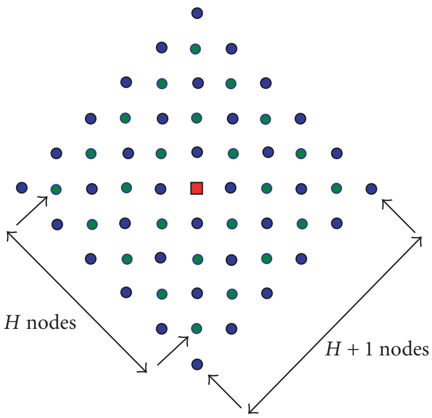

Figure 7: Total nodes within L_1 distance $\widehat{H}$ is $2\widehat{H}^2 + 2\widehat{H}$.

Upper bound

Before stating the upper bound on the uniform throughput, let us define $S_{\max}$ as the number of nodes that are vwi to a given node. This number depends on the beamwidth only. Let α' be an upper bound[8] on the fraction of nodes that are vwi for a given node in the zero beamwidth case.

Lemma 1. *In terms of the maximum hop size,*

$$S_{\max} = 2\alpha' \widehat{H}(\widehat{H} + 1). \tag{27}$$

Proof. For a system with nodes on a grid, it is obvious that the maximum number of nodes within a distance $\widehat{H}$ can be found by counting the nodes within two squares of sides $\widehat{H}$ and $\widehat{H} + 1$ surrounding the source. This is shown in Figure 7 for an $\widehat{H} = 5$. The total number of nodes within these squares is $(\widehat{H} + 1)^2 + \widehat{H}^2 = 2\widehat{H}^2 + 2\widehat{H} + 1$. Removing the source node itself from the count results in a total of $2\widehat{H}^2 + 2\widehat{H}$ nodes within a distance $\widehat{H}$. Multiplying by the maximum fraction of nodes which are vwi α' yields the statement of the lemma. □

We can now obtain an upper bound on the throughput.

Theorem 7. *The uniform throughput $\mathcal{T}$ satisfies the inequality*

$$\mathcal{T} \leq \min\left\{\frac{\alpha' n}{2}, \frac{S_{\max}}{2}, \frac{g}{2}, \frac{g\widehat{H}}{\sqrt{n}}, \frac{S_{\max}\widehat{H}}{\sqrt{n}}\right\}. \tag{28}$$

Proof. We know from Theorem 1 that $\mathcal{T} \leq s/\overline{h}$, where $\overline{h}$ is the average length of the shortest source-to-destination path (measured in hops). It is clear that $s \leq \min\{g, S_{\max}, \alpha' n\}$. Also, for any transmission scheme, $\overline{h} \geq \max\{2, \sqrt{n}/\widehat{H}\}$ (the latter is because the distance between sources and destinations is equal to m, and the longest possible hop is equal to $\widehat{H}$). Noting that $\sqrt{n}/\widehat{H} > 2$ implies $\alpha' n > S_{\max}$, we obtain the statement of the theorem. □

[8] We can set, for example, $\alpha' = 0.72$ which is valid for $m \geq 5$ (see Figure 2).

In case $\sqrt{n}/\widehat{H} > 2$, that is, when it takes more than two hops to reach the destination from the corresponding source, the upper bound of Theorem 7 can be further tightened.

Theorem 8. *If $\sqrt{n}/\widehat{H} > 2$, the uniform throughput can be upper bounded as*

$$\mathcal{T} \leq \min\left\{\frac{g\overline{l}'}{\sqrt{n}}, \frac{S_{\max}\overline{l}'}{\sqrt{n}}\right\}. \tag{29}$$

Proof. It follows from Theorem 2 by noting that all source-destination distances are equal to $m = \sqrt{n}$, and therefore $\overline{L} = \sqrt{n}$. □

Achievability

Let us assume, without loss of generality,[9] that $g < S_{\max}$ and $g < \alpha' n$. Under these assumptions, the upper bounds of Theorems 7 and 8 take the form

$$\mathcal{T} \leq \min\left\{\frac{g}{2}, \frac{g\overline{l}'}{\sqrt{n}}\right\}. \tag{30}$$

Let us consider the two cases separately.

Case $\overline{l}'/\sqrt{n} < 1/2$

Consider the following transmission scheme.

Transmission Scheme 1

In this transmission scheme, the node-to-node transmissions are always to vwi nodes. The g successful transmissions from each node are possible in every time slot. Let us denote the possible hops to vwi nodes by the corresponding length in horizontal and vertical directions. Thus, if a transmission to a vwi node can be made in which a packet moves by k lattice space in horizontal direction and by l lattice space in the vertical direction, we denote such hop by (k, l). Because of system symmetry (no boundaries), all nodes have the same vwi hops available to them. Due to lattice symmetry, for every (k, l) vwi hop there is $\operatorname{sgn}(lk)(l, k)$ vwi hop.[10] Let us assume, for additional simplicity, that the number of beam g is divisible[11] by 4.

Let $(k_1, l_1), (l_1, k_1), \ldots, (k_{g/2}, l_{g/2})$ be g hops listed from the largest value of the hop length $|k| + |l|$ in a nonincreasing order. Our goal is to construct g paths from a source to the corresponding destination in such a way that

(1) each node in the network is able to transmit to g vwi nodes in every time slot;
(2) each node uses the same g hops in a given time slot—this ensures that every node receives exactly g transmissions.

[9] The other cases can be considered in an analogous way.
[10] For example, the availability of $(\widehat{H} - 1, -1)$ hop to a vwi node implies that the hop $(1, -(\widehat{H} - 1))$ also leads to a vwi node.
[11] If this is not so, corresponding modifications can easily be made.

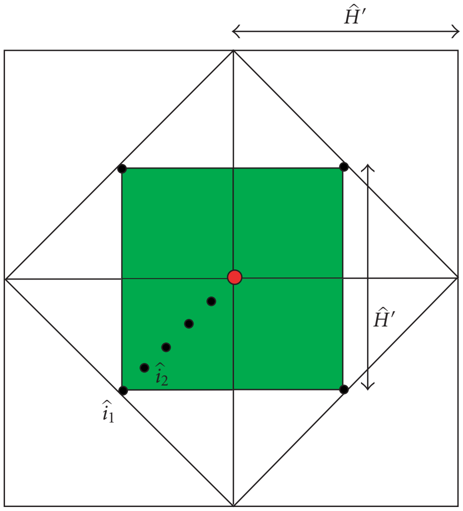

Figure 8: All packets will arrive in the diagonal nodes within the center $\hat{H} \times \hat{H}$ square, $\mathcal{S}_i$.

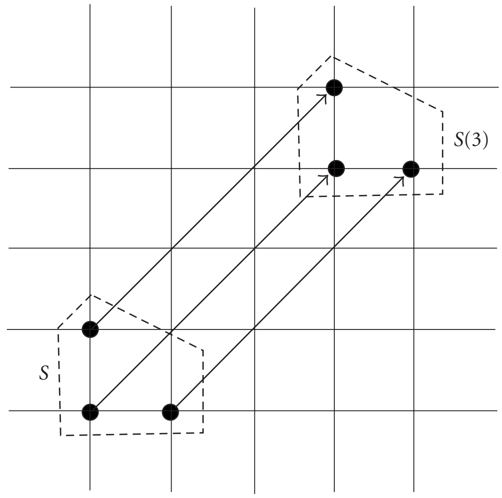

Figure 9: Sets S and "shifted" $S(3)$.

It is easy to see that in order to satisfy the first condition above, it is sufficient to demand that

(i) the first hop directions of all g paths are different;

(ii) each hop direction has a *unique predecessor*: a hop (k_i, l_i) can only follow a hop (k_j, l_j) for a unique value j.

Phase I

We can satisfy these demands by constructing paths from pairs of directions. Namely, Let path 1 consist of hops (k_1, l_1) and $\operatorname{sgn}(k_1 l_1)(l_1, k_1)$ following each other: $P_1 = \{(k_1, l_1), \operatorname{sgn}(k_1 l_1)(l_1, k_1), (k_1, l_1), \dots\}$. Path 2 will consist of the same hops with odd and even hops exchanged: $P_2 = \{\operatorname{sgn}(k_1 l_1)(l_1, k_1), (k_1, l_1), \operatorname{sgn}(k_1 l_1)(l_1, k_1), \dots\}$. Paths 3 and 4 are constructed in the same way from hops (k_2, l_2) and $\operatorname{sgn}(k_2 l_2)(l_2, k_2)$ and so on.

It is easy to see that the paths constructed in the above way generically will not necessarily end up exactly at the destination. On the other hand, since each packet will move by $m \leq \hat{H}$ lattice spaces in both horizontal and vertical directions after any two successive hops, it will eventually arrive at one of diagonal nodes within the $\hat{H}' \times \hat{H}'$ square[12] surrounding the destination. We denote such a square around a destination node i by $\mathcal{S}_i$ (see Figure 8).

It remains to complete the paths $P_1, P_2, \dots, P_g$ so that they end up exactly at the destination. We should also do it while maintaining the property of unique predecessor. In order to make this possible, we have to find a different (i.e., using different hops) continuation for every path P_j, $j = 1, 2, \dots, g$. As we see from the above paths P_j, different values of j may end up at different diagonal nodes within the square $\mathcal{S}_i$. Because of the symmetry between the four quadrants, it is sufficient to find distinct continuations of paths $P_1, \dots, P_{g/4}$.

Let us introduce some additional notation.

(i) If V_i and V_j are sets of notes that are vwi to nodes i and j, respectively. We will denote by V_{ij} the set of nodes that are vwi to both nodes i and j, or $V_{ij} \equiv V_i \cap V_j$.

(ii) If S is any set of nodes (lattice sites), we will denote by $S(l)$ the set of nodes that is obtained by shifting the nodes in the set S by l lattice spaces in both horizontal and vertical directions (see Figure 9).

(iii) We also introduce special notation for the diagonal nodes within the square $\mathcal{S}_i$. We denote by $\hat{i}_1$ the diagonal node in the "left-bottom corner" of $\mathcal{S}_i$, by $\hat{i}_2$ the next diagonal node in the direction of $d(i)$, and so on (see Figure 8).

As mentioned previously, Phase I of Transmission Scheme 1 ends with packets arriving at diagonal nodes of square $\mathcal{S}_i$. Suppose the total number of such packets waiting at nodes $\hat{i}_1, \hat{i}_2, \dots$ is $n_1, n_2, \dots$, respectively. We would like to find whether it is possible to find distinct 2 hop paths for all these packets to $d(i)$. Consider the following algorithm.

Algorithm 1. (1) *Let $k_1, k_2, \dots, k_r$ be values of the index l such that $n_l > 0$.*

(2) *If $n_{k_1} > |V_{\hat{i}_{k_1} d(i)}|$, stop. Finding the required path continuations is impossible.*

(3) *Otherwise, choose a set of nodes $S_1^{(i)} \subseteq V_{\hat{i}_{k_1} d(i)}$ so that $|S_1^{(i)}| = n_{k_1}$ and $|V_{\hat{i}_{k_2} d(i)} \setminus (S_1^{(i)} \cup S_1^{(i)}(k_2 - k_1))|$ is maximized.*

(4) *If $n_{k_2} > |V_{\hat{i}_{k_2} d(i)} \setminus (S_1^{(i)} \cup S_1^{(i)}(k_2 - k_1))|$, stop. Finding the required path continuations is impossible.*

[12] Here $\hat{H}'$ is equal to $\hat{H}$ if $\hat{H}$ is odd and to $\hat{H} + 1$ if $\hat{H}$ is even.

TABLE 1: Cardinalities of sets $V_{\hat{i}_l d(i)}$, for $D = 25^\circ$, $D = 15^\circ$, $D = 10^\circ$, and $D = 5^\circ$, respectively.

Node	Cardinality	
$\hat{i}_1$	11	$D = 25^\circ$
$\hat{i}_2$	8	
$\hat{i}_1$	23	$D = 15^\circ$
$\hat{i}_2$	18	
$\hat{i}_1$	24	$D = 10^\circ$
$\hat{i}_2$	39	
$\hat{i}_3$	24	
$\hat{i}_1$	52	$D = 5^\circ$
$\hat{i}_2$	42	
$\hat{i}_3$	64	
$\hat{i}_4$	50	
$\hat{i}_5$	89	
$\hat{i}_6$	54	

(5) *Otherwise, choose sets of nodes*[13] $S_1^{(i)} \subseteq V_{\hat{i}_{k_1} d(i)}$ *and* $S_2^{(i)} \subseteq V_{\hat{i}_{k_2} d(i)}$ *so that* $|S_1^{(i)}| = n_{k_1}$, $|S_2^{(i)}| = n_{k_2}$ *and* $|V_{\hat{i}_{k_3} d(i)} \setminus (S_1^{(i)} \cup S_1^{(i)}(k_2 - k_1) \cup S_2^{(i)} \cup S_2^{(i)}(k_3 - k_2))|$ *is maximized.*

(6) *Continue in the same way until either the required path continuations are found or declared to be impossible to find. In the former case the output of the algorithm will include sets* $S_1^{(i)}, \ldots, S_k^{(i)}$.

The cardinalities of sets $V_{\hat{i}d(i)}$ for different values of D are shown in Table 1.

Thus we arrive at the second phase of source-to-destination transmission.

Phase II

Forward the packets waiting at node $\hat{i}_{k_j}$, $j = 1, 2, \ldots, r$, to the destination via two hops: from $\hat{i}_{k_j}$ to one of the nodes in the set $S_j^{(i)}$ and from that node to the destination.

We can now state the achievability result. Let $l_{\min}$ be the smallest L_1 hop length used in Phase I of Transmission Scheme 1.

Theorem 9. *The uniform throughput of at least* $g/\lfloor \sqrt{n}/l_{\min} \rfloor + 2$ *is achievable provided* $g \leq |V_i|$ *and g path continuations can be found using Algorithm 1.*

Proof. The proof follows directly from Theorem 3 where we set $r = g$ and $h_{\max} = \lfloor \sqrt{n}/l_{\min} \rfloor + 2$. It only remains to be noted that in order to make the overall transmission strategy (Transmission Scheme 1) satisfy the conditions of Theorem 3, we need to synchronize Phase I and Phase II. Namely, out of every $\lfloor \sqrt{n}/l_{\min} \rfloor + 2$ time slots, $\lfloor \sqrt{n}/l_{\min} \rfloor$ are dedicated to all nodes performing Phase I and 2 time slots to all nodes performing Phase II. □

[13] Note that the set $S_1^{(i)}$ chosen at this step may be different from $S_1^{(i)}$ chosen at the previous step.

For some specific values of g and D we can actually establish the feasibility of Algorithm 1 and make more specific claims. For example, if $g = 8$, it is easy to see from Table 1 that finding the required path continuations are possible for any $D \leq 10^\circ$. A brief consideration of the worst case scenario also shows that this is true for $g = 16$ as well. For larger values of g closer inspection would be needed. We can formulate these observations as a corollary. Let $\hat{H} = 2 + \lfloor 1/\tan D \rfloor$ as shown in Theorem 6.

Corollary 2. *The uniform throughput of at least* $g/(\lfloor \sqrt{n}/\hat{H} \rfloor + 2)$ *for* $g \leq 8$ *and* $D \leq 25^\circ$ *is achievable.*

Case $\bar{l}'/\sqrt{n} > 1/2$

In this case, the inequality $g\hat{H}/\sqrt{n} > 1/2$ holds as well, and, therefore, the source for every destination is located within the corresponding square $\mathcal{S}_i$. As we have already seen, two-hop transmission is possible under these conditions. We have the following theorem.

Theorem 10. *The uniform throughput of at least* $g/2$ *is achievable provided* $V_{id(i)} \geq g$.

3.4. Numerical results

The upper and lower bounds on uniform throughput per node for a system with beamwidth 25, 15, 10, and 5 degrees were numerically computed. The number of beams is set to $g = 8$. The throughput is measured in terms of W. Each beam is capable of sending W bits of data per second. The results are shown in Figures 10, 11, 12, and 13.

These figures also show the size of the network required to bring the throughput per node down to one W, where there is no longer any spatial multiplexing gain. The results show that the network can be very large (thousands of nodes even for 10 degree beamwidth) before this occurs. Therefore, using spatial multiplexing with directional antennas, ad hoc networking can be implemented in practical sized systems without experiencing performance degradation (compared with the individual link rate) even for fairly wide beamwidths.

4. CONCLUSION AND DISCUSSION

We analyzed the throughput of ad hoc networks with nodes located on a square lattice with periodic boundary conditions. For the case of infinitely narrow beamwidth, we found that a uniform throughput proportional to the maximum number of beams a node can form and independent of the system size is achievable.

We also showed large gains (compared to systems without spatial multiplexing) for a practical system with a small number of antennas and a finite practically achievable beamwidth. These gains have been shown to offset the interference effect on throughput up to network sizes in the

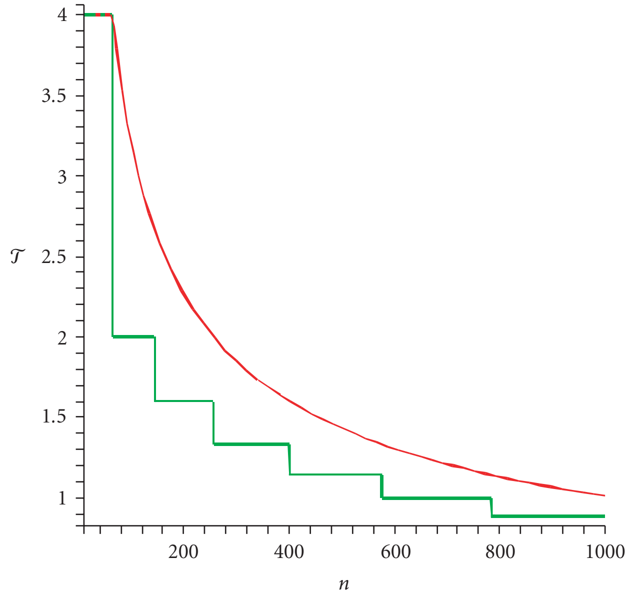

Figure 10: Throughput $\mathcal{T}$ versus n (upper and lower bounds) for beamwidth of 25 degrees.

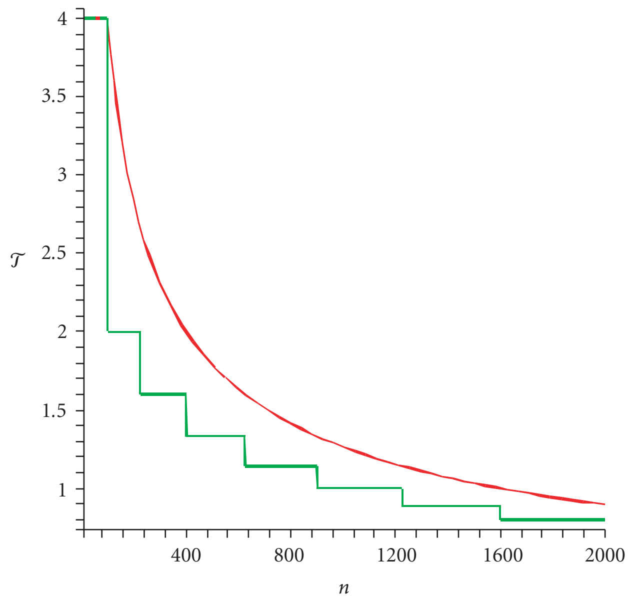

Figure 11: Throughput $\mathcal{T}$ versus n (upper and lower bounds) for beamwidth of 15 degrees.

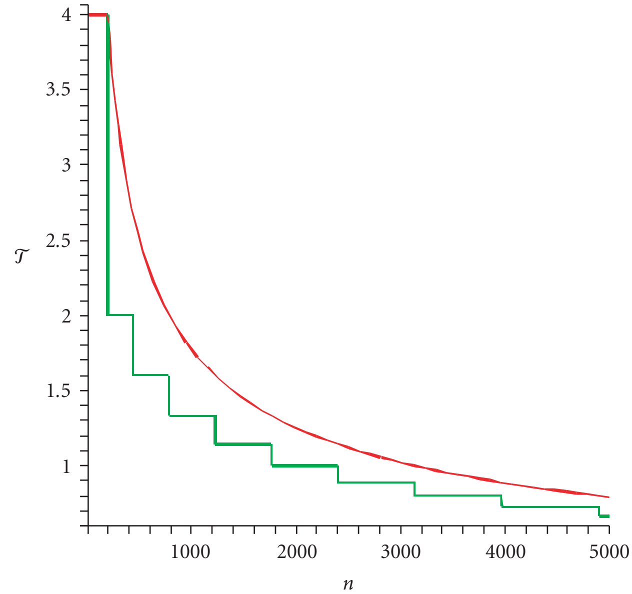

Figure 12: Throughput $\mathcal{T}$ versus n (upper and lower bounds) for beamwidth of 10 degrees.

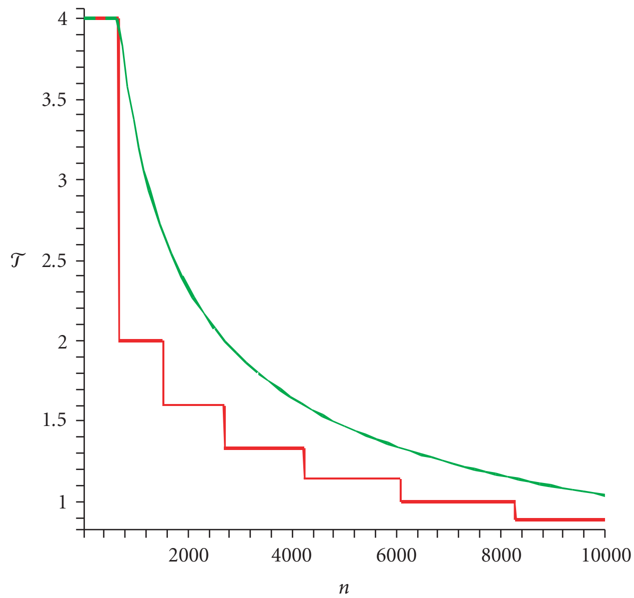

Figure 13: Throughput $\mathcal{T}$ versus n (upper and lower bounds) for beamwidth of 5 degrees.

thousands thus making network of such sizes effectively unaffected by interference-related throughput degradation.

Our results demonstrate that there is a strong incentive to design and deploy ad hoc networks with good directional antenna or beamforming capability in order to improve capacity or simplify the communication protocol design.

Obviously, one of the limitations of the proposed analysis approach is the assumption that the nodes are located on a regular square grid. While this assumption makes the analysis tractable, it does not fully reflect the topology of the majority of real networks. While the full consideration of more realistic models goes well beyond the scope of this paper, we will attempt to sketch an argument showing that the main results would likely not change much under a more realistic model.

The main result of the paper depends on the ability to simultaneously transmit along g beams over long distances (longer than the typical internode distance). To consider a different, perhaps more realistic model, let us assume that the nodes are placed randomly with a unit density inside a circle

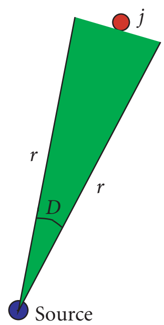

FIGURE 14: The sector of area $r^2D/2$ has to contain no nodes for the node j at a distance r from the source to be vwi to the latter.

(or square) of area n^2. The probability that a node at a distance r from a source is vwi to it can be approximately calculated (using the two-dimensional Poisson distribution) as $e^{-r^2D/2}$ (see Figure 14). The total number of nodes at a distance r from the source can be approximately calculated as $2\pi r$ (since the node density is unity). Therefore, the number of nodes at a distance r from the source that are vwi to it can be estimated as $2\pi r e^{-r^2D/2}$.

So for $g = 8$, one should be able to find 8 relays at a distance $r \approx 6$ from the source for $D = 5°$. For $D = 10°$, one should be able to find 8 relays at a distance of $r \approx 3$, and for $D = 15°$, 8 relay nodes can be found at a distance of $r \approx 2$. We see that these distances are roughly 2 times smaller than those found in this paper for the square grid model. This would result in the "efficient" network size (the size for which the throughput is no less than W) roughly 4 times smaller. Or, equivalently, the throughput for the same network size would be roughly 2 times smaller than the one found in this paper. It is also fairly easy to see that the total number of vwi nodes for both models is about the same for the same value of D, but the regular grid model has an advantage in the largest hop distance because of the presence of "preferred" directions along the grid's main axes. Also note that the real-life networks may well have such directions (i.e., along major streets in urban networks) and this would most likely lead to values of the achievable throughput between those found in this paper and those that can be obtained via a careful analysis of a random node placement model.

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