Hindawi Publishing Corporation
EURASIP Journal on Wireless Communications and Networking
Volume 2008, Article ID 186020, 9 pages
doi:10.1155/2008/186020

Research Article

Statistical Analysis of Multipath Fading Channels Using Generalizations of Shot Noise

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Received 3 December 2007; Revised 30 March 2008; Accepted 22 July 2008

Recommended by Xueshi Yang

This paper provides a connection between the shot-noise analysis of Rice and the statistical analysis of multipath fading wireless channels when the received signals are a low-pass signal and a bandpass signal. Under certain conditions, explicit expressions are obtained for autocorrelation functions, power spectral densities, and moment-generating functions. In addition, a central limit theorem is derived identifying the mean and covariance of the received signals, which is a generalization of Campbell's theorem. The results are easily applicable to transmitted signals which are random and to CDMA signals.

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1. INTRODUCTION

A statistical temporal model which captures the time-varying and time-spreading properties of the channel is the so-called multipath fading channel model (MFC) [1, pages 12, 13, 760, 761], [2, page 146], [3]. The output of such channel, when the input is the low-pass signal $x_\ell(t)$, is given by

$$y_\ell(t) = \sum_{i=1}^{N(t)} r_i(\tau_i) e^{j\Phi_i(t,\tau_i)} x_\ell(t-\tau_i), \tag{1}$$

which corresponds to that of the so-called quasistatic channel. Here, $r_i(\tau)$, $\Phi_i(t,\tau)$, τ_i denote the attenuation, phase, and propagation time delay, respectively, of the signal received in the ith path, and $N(t)$ denotes the number of paths at time t. The phase $\Phi_i(t,\tau)$ is typically a function of the carrier frequency, the relative velocity between the transmitter and the receiver, and the angle of arrivals and phase of the incident on the receiver plane wave [4–6]. On the other hand, if $x_\ell(t)$ is the low-pass equivalent representation of a bandpass signal, modulated at some carrier frequency ω_c, namely, $x(t) = \mathrm{Re}\{x_\ell(t)e^{j\omega_c t}\}$, then the received bandpass signal is $y(t) = \mathrm{Re}\{y_\ell(t)e^{j\omega_c t}\}$. In the works found in the literature, the authors often omit this explicit dependence of r_i on τ_i, during the computation of the various statistics ([2, page 146] is an exception). Although for a deterministic or fixed sample path of $\{N(s);\ 0 \le s \le t\}$ the computation of the statistical properties of $y_\ell(t)$ is not affected by this omission, this is not the case when the ensemble statistics are analyzed. Ensemble statistics using a counting process as simple as the nonhomogeneous Poisson process reveal an additional smoothing property associated with each propagation environment, which is expressed in terms of the rate of the Poisson process and the attenuations.

The objective of this paper is to introduce a unified framework for computing the statistical properties of the received signal when $\{\tau_i\}_{i\ge1}$ are the points of a Poisson counting process $N(t)$, while for fixed sample paths of the points the distribution of the instantaneous amplitude and phase, $\{r_i(\tau_i), \Phi_i(t,\tau_i)\}$, $i = 1,2,\ldots,$ is arbitrary, by performing an analysis which can be viewed as a generalization of the shot-noise analysis investigated by Rice [7, 8] in the mid 1940's. This approach is similar to the one considered in [9] which investigates the statisticalproperties of cochannel interference. However, in [9] the authors are interested in stable distributed processes although their approach can be extended to other distributions.

In [10, 11], the authors questioned the accuracy of the Poisson counting processes in matching experimental data of path arrival time and number of paths, and thus a modified

Poisson process is introduced, the so-called $\Delta - K$ model. However, the failure of the Poisson process to model path arrival times does not imply that the Poisson model will also be inappropriate when considered as part of (1) to study the statistics of the received signal. In this paper, we show that when the Poisson counting process is included in (1), then various existing properties of MFCs, such as the power delay profile, the Doppler spread, and the Gaussianity of the channel, are predicted. Due to its simplicity, the Poisson counting process is the most natural process to start the analysis with. It can form the core for subsequent generalizations in which the rate of the counting process is random. The validity of the Poisson counting process is illustrated through subsequent calculations of second-order statistics of $y_\ell(t)$ and $y(t)$, their power spectrum densities, and their moment-generating functions, which reveals that when the rate of the Poisson process is sufficiently high, the received signal is normally distributed with mean and covariance functions being identified. On the other hand, when the rate of the Poisson process is low, the received signal can no longer be assumed as normally distributed. In the latter case, the probability that the individual paths overlap is negligible, while in the former case this probability is quite high.

The above analysis is important when designing specific receivers as follows. Assume that (1) represents the baseband received signal which is corrupted by additive white Gaussian noise. A well-known optimal receiver is the matched filter, which maximizes the output signal-to-noise ratio [1]. The implementation of the matched filter requires the knowledge of the power spectral density of (1), which is computed in the paper. Moreover, in many applications such as filter design and interference analysis, it is important to know the precise joint distribution of the processes $(\{y_l(t)\}_{t\geq 0}, \{y(t)\}_{t\geq 0})$. This joint distribution is also computed when $\{y_l(t)\}_{t\geq 0}$, $\{y(t)\}_{t\geq 0}$ are Gaussian distributed. Moreover, the results of the paper when combined with [9] can be used to analyze interference statistics of multipath fading channels.

The paper is organized as follows. Section 2 discusses correlation properties and relations to known statistical properties of $y_\ell(t)$ and $y(t)$. Section 3 presents several power spectral densities of $y_\ell(t)$ and $y(t)$ for any information signal. Section 4 establishes central limit theorems which imply Gaussianity of $y_\ell(t)$ and $y(t)$.

Notation 1. $\mathbb{N}_+$ denotes the set of positive integers; E will denote the expectation operator; $|c|^2 \triangleq c^\star c$, where $c \in \mathcal{C}$ is complex and "$\star$" denotes complex conjugation. For $\mathcal{T} \in \mathcal{L}(\mathcal{C}^m; \mathcal{C}^n)$, a linear operator $\mathcal{T}^\dagger$ denotes Hermitian conjugation. For $\rho \in \mathcal{C}^n$, where $\rho_{R_i} \triangleq \mathrm{Re}(\rho_i)$ and $\rho_{I_i} \triangleq \mathrm{Im}(\rho_i)$, $1 \leq i \leq n$, denote the real and imaginary components of ρ, respectively. The complex derivatives with respect to ρ and $\rho^\star$ are defined in terms of real derivatives as follows: $\partial/\partial\rho_i \triangleq (\partial/\partial\rho_{R_i} - j(\partial/\partial\rho_{I_i}))/2$, $\partial/\partial\rho_i^\star \triangleq (\partial/\partial\rho_{R_i} + j(\partial/\partial\rho_{I_i}))/2$, $1 \leq i \leq n$. For f, g real- or complex-valued functions, $f * g$ denotes convolution operation of f with g, and $\mathcal{F}_\tau\{f\}$ denotes Fourier transform (FT).

2. MEAN, VARIANCE, AND CORRELATION

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space equipped with filtration $\{\mathcal{A}_t\}_{t\geq 0}$ and finite-time $[0, T_s]$, $T_s < \infty$, on which the following random variables are defined: $r_i : \Omega^r \times \Omega^\tau \to \mathfrak{R}$, $\phi_i : \Omega^\phi \to \mathfrak{R}$, $\tau_i : \Omega^\tau \to \mathfrak{R}$, $\omega_{d_i} : \Omega^{\omega_d} \to \mathfrak{R}$, $N : [0, T_s) \times \Omega \to \mathbb{N}_+$, $\mathbf{m}_i(\tau_i) \triangleq (r_i(\tau_i), \phi_i, \omega_{d_i})$. This paper investigates the statistical properties of a noncausal version of (1), namely,

$$\begin{aligned} y_\ell(t) &= \sum_{i=1}^{N(T_s)} r_i(\tau_i) e^{j\phi_i} e^{-j(\omega_c+\omega_{d_i})\tau_i + j\omega_{d_i} t} x_\ell(t-\tau_i) \\ &\triangleq \sum_{i=1}^{N(T_s)} h_\ell(t, \tau_i; \mathbf{m}_i(\tau_i)), \end{aligned} \tag{2}$$

where $0 \leq t \leq T_s$ and its bandpass representation is

$$\begin{aligned} y(t) &= \mathrm{Re}\left\{\left[\sum_{i=1}^{N(T_s)} r_i(\tau_i) e^{j\phi_i} e^{-j(\omega_c+\omega_{d_i})\tau_i + j\omega_{d_i} t} x_\ell(t-\tau_i)\right] e^{j\omega_c t}\right\} \\ &\triangleq \mathrm{Re}\left\{\sum_{i=1}^{N(T_s)} h(t, \tau_i; \mathbf{m}_i(\tau_i)) e^{j\omega_c t}\right\} \\ &= \mathrm{Re}\left\{\sum_{i=1}^{N(T_s)} h(t-\tau_i; \mathbf{m}_i(\tau_i)) e^{j\omega_c t}\right\}, \end{aligned} \tag{3}$$

in which $h_\ell(t, \tau_i; \mathbf{m}_i(\tau_i)) = r_i(\tau_i) e^{j\phi_i} e^{-j\omega_c\tau_i + j\omega_{d_i}(t-\tau_i)} x_\ell(t-\tau_i)$, $h(t, \tau_i; \mathbf{m}_i(\tau_i)) = r_i(\tau_i) x(t-\tau_i)$, $x(t) = \mathrm{Re}\{x_\ell(t) e^{j(\omega_c t + \omega_{d_i} t + \phi_i)}\}$, r_i is the attenuation, τ_i is the time delay, ϕ_i is the phase, ω_{d_i} is the Doppler spread of the ith path, and ω_c is the carrier frequency. For fixed $\tau_i = \tau$, the dependence of the attenuations $\{r_i(\tau)\}_{i\geq 1}$ on τ implies that the attenuations are random variables. Notice that each occurrence time τ_i is associated with $\mathbf{m}_i(\tau_i) = (r_i(\tau_i), \phi_i, \omega_{d_i})$, and $h(t, \tau_i; \mathbf{m}_i(\tau_i))$ (or $h_\ell(t, \tau_i; \mathbf{m}_i(\tau_i))$ may be viewed as the impulse response at time t due to the occurrence of τ_i. In the preliminary calculations, it is assumed that for a fixed occurrence time $\tau_i = \tau$, $\{h_\ell(t; \tau; \mathbf{m}_i(\tau)\}_{t\geq 0}$ and $\{h(t; \tau; \mathbf{m}_i(\tau)\}_{t\geq 0}$, $i = 1, 2, \ldots$, are independent of the counting process $N(T_s)$. However, in obtaining explicit expressions, we will often make the following assumption.

Assumption 1. Let $\{\lambda_T(s) \triangleq \lambda \times \lambda_c(s);\ 0 \leq s \leq t\}$ denote the nonnegative and nonrandom rate of the counting process $\{N(s);\ 0 \leq s \leq t\}$, where λ is constant and nonrandom and $\lambda_c(t)$ is a time-varying nonrandom function. For fixed $\tau_i = \tau$, the random processes $\{h(t, \tau; \mathbf{m}_i(\tau)\}_{t\geq 0}$ (resp., $\{h_\ell(t, \tau; \mathbf{m}_i(\tau)\}_{t\geq 0}$), $i = 1, 2, 3, \ldots$, are mutually independent and identically distributed, having the same distribution as $\{h(t, \tau; \mathbf{m}(\tau)\}_{t\geq 0}$ (resp., $\{h_\ell(t, \tau; \mathbf{m}(\tau)\}_{t\geq 0}$), and are also independent of $\{N(s);\ 0 \leq s \leq t\}$.

Assumption 1 is invoked only when seeking closed-form expressions for various statistics. We note that when λ is a random variable, most of the subsequent results of this note remain valid provided that we include an extra integration with respect to the density of λ. Such generalizations do not suffer from the orderliness and the independent increment properties of the Poisson counting process; however, the analysis is more complicated and should be discussed elsewhere.

Mean and variance

The mean (expected value) and the variance of the received complex signal $y_\ell(t)$ are, respectively, defined by $\overline{y_\ell}(t) \triangleq E[\sum_{i=1}^{N(T_s)} h_\ell(t,\tau_i;\mathbf{m}_i(\tau_i))]$ and $\mathrm{Var}(y_\ell(t)) \triangleq E[y_\ell^\star(t)y_\ell(t)] - \overline{y_\ell^\star}(t)\overline{y_\ell}(t)$, where $E[\cdot]$ denotes expectation with respect to the joint density of $\{\mathbf{m}_i(\tau_i), N(T_s), \tau_i\}_{i\geq 1}$. Suppose that $\{N(s); 0 \leq s \leq T_s\}$ is Poisson with rate $\lambda_T(t) \geq 0$, for all $t \in [0,T_s]$. Under the assumption that $\{\mathbf{m}_i(\tau)\}_{i\geq 1}$ are independent of $N(T_s)$ and conditioning on $N(T_s)=k$, the delay times $\{\tau_i\}_{i=1}^k$ are independent identically distributed with density $f(t) = \lambda_T(t)/\int_0^{T_s}\lambda_T(t)dt$, $0 \leq t \leq T_s$ (see [12]). Hence, $\overline{y_{\ell,k}}(t) \triangleq E[\sum_{i=1}^{N(T_s)} h_\ell(t,\tau_i;\mathbf{m}_i(\tau_i)) \mid N(T_s)=k] = \sum_{i=1}^k \int_0^{T_s} f(\tau)E[h_\ell(t,\tau;\mathbf{m}_i(\tau))]d\tau$. Clearly, if the number of paths during $[0,T_s]$ is known, $\overline{y_{\ell,k}}(t)$ gives the average received instantaneous signal. However, this is usually unknown unless one sounds the channel assuming a low noise level; its ensemble average is obtained from $E[y_\ell(t)] = \sum_{k=1}^\infty \overline{y_{\ell,k}}(t)\mathrm{Prob}\{N(T_s)=k\}$. Similarly, we compute $E[|y_\ell(t)|^2] = \sum_{k=1}^\infty \overline{y_{\ell,k}}^2(t)\mathrm{Prob}\{N(T_s)=k\}$ and the variance, where

$$\begin{aligned}
\overline{y_{\ell,k}^2}(t) &\triangleq E[y_\ell^2(t) \mid N(T_s)=k] \\
&= \sum_{i=1}^k \int_0^{T_s} f(\tau)E[|h_\ell(t,\tau;\mathbf{m}_i(\tau))|^2]d\tau \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^k \int_0^{T_s} f(\tau_i)d\tau_i \int_0^{T_s} f(\tau_j)d\tau_j \\
&\qquad \times E[h_\ell^\star(t,\tau_i;\mathbf{m}_i(\tau_i))h_\ell(t,\tau_j;\mathbf{m}_j(\tau_j))],
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\mathrm{Var}(y_\ell(t)) \\
&= \sum_{k=1}^\infty \overline{y_{\ell,k}^2}(t)\mathrm{Prob}(N(T_s)=k) - (\overline{y_\ell}(t))^2 \\
&= \sum_{k=1}^\infty \mathrm{Prob}(N(T_s)=k) \\
&\quad \times \sum_{i=1}^k \bigg\{ \int_0^{T_s} f(\tau)E[r_i^2(\tau)]|x_\ell(t-\tau)|^2 d\tau \\
&\qquad - \left| \int_0^{T_s} f(\tau)E[r_i(\tau)e^{j\phi_i + j\omega_{d_i}(t-\tau) - j\omega_c\tau}x_\ell(t-\tau)]d\tau \right|^2 \bigg\}, \\
&\qquad\qquad \text{if } h(t,\tau;\mathbf{m}_i(\tau)) \text{ is uncorrelated.}
\end{aligned} \tag{5}$$

In practice, there exists a finite k such that $\mathrm{Prob}(N(T_s)=n)$ is small for $n \geq k$; in which case, the infinite series can be approximated by a finite series, and thus (4) and (5) can be computed. Alternatively, if the conditions of Assumption 1 are satisfied, which is sufficient to assume that $\{r_i(\tau), \phi_i, \omega_{d_i}\}$, for all $i \in \mathbb{N}_+$, are mutually independent and identically distributed, independently of the random process $\{N(t); 0 \leq s \leq T_s\}$, then an explicit closed-form expression is given in the next lemma, which is a generalization of the shot-noise effect discussed by Rice in [7, 8].

Lemma 1. *Consider model (2)-(3) under Assumption 1. Then,*

$$\begin{aligned}
&E[y_\ell(t)] \\
&\quad = \int_0^{T_s} \lambda_T(\tau)E[h_\ell(t,\tau;\mathbf{m}(\tau))]d\tau \\
&\quad = \int_0^{T_s} \lambda_T(\tau)E[r(\tau)e^{j\phi + j\omega_d(t-\tau)}e^{-j\omega_c\tau}]x_\ell(t-\tau)d\tau,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathrm{Var}(y_\ell(t)) &= \int_0^{T_s} \lambda_T(\tau)E[|h_\ell(t,\tau;\mathbf{m}(\tau))|^2]d\tau \\
&= \int_0^{T_s} \lambda_T(\tau)E[r^2(\tau)]|x_\ell(t-\tau)|^2 d\tau,
\end{aligned} \tag{7}$$

for $0 \leq t \leq T_s$.

Remark 1. Some observations concerning the results of Lemma 1 are now in order. These observations are important because they provide additional insight regarding the role of the rate of Poisson process in modeling quasistatic channels.

(1) Clearly, the rate of the Poisson process is an important parameter which shapes the statistics of the received signal, and therefore the multipath delay profile and the Doppler spread. It models the filtering properties of the propagation environment. If the arrival times of the different paths are known (information which is obtained by sounding the channel), then the rate of the Poisson process should be replaced by a linear combination of impulses. Thus, by setting $\lambda_T(t) = \sum_{i=1}^N \lambda_i\delta(t-\tau_i)$, we obtain

$$\begin{aligned}
\mathrm{Var}(y_\ell(t)) &= \int_0^{T_s} \sum_{i=1}^N \lambda_i\delta(t-\tau_i)E[|h_\ell(t,\tau;\mathbf{m})|^2]d\tau \\
&= \sum_{i=1}^N \lambda_i E[r^2(\tau_i)]|x_\ell(t-\tau_i)|^2,
\end{aligned} \tag{8}$$

for $0 \leq t \leq T_s$, which is exactly what one would obtain if the arrival times of the multipath components are known.

(2) *Tapped delay channel.* Consider the tapped delay channel model, which corresponds to a frequency-selective channel with transmitted signal bandwidth W which is greater than the coherence bandwidth B_{coh} of the channel, and $W \gg B_{\mathrm{coh}}$. In this case, the sampling theorem (see [1, pages 795–797]) leads to the tapped delay line model, where $N = [(1/B_{\mathrm{coh}})W] + 1$, $\tau_i = i/W$, $1 \leq i \leq N$, and N is the number of resolvable paths. This tapped delay model can be generated from the model presented using a Poisson process by choosing the rate of the Poisson process

so that most points are concentrated at $\{i/W\}_{i\geq 1}$ (e.g., letting $\lambda_T(t)$ be a series of mountains concentrated near i/W). That is, the orderliness effect of the Poisson process is mitigated because of the limitations of the equipment that is used to measure the received signal. In the next two statements, we present a comparison of the computation of the received power when the arrival times of the multipath components are known and when these are assumed to be the points of a homogeneous Poisson process.

(3) *Wideband transmission.* Consider the periodic transmission of a pulse $x_\ell(t) = \pi(t)$ every T_s seconds, where $\pi(t) = \sqrt{\overline{\tau}_m/T_c}$ if $0 \leq t \leq T_c$ and $\pi(t) = 0$ or, otherwise, where $T_s \gg \overline{\tau}_m$, with $\overline{\tau}_m$ denoting the duration of the channel impulse response (e.g., excess delay of the channel).

Suppose that the low-pass received signal is

$$y_{\ell,N}(t) = \sum_{i=1}^{N} r_i e^{j\phi_i} e^{-j(\omega_c+\omega_{d_i})\tau_i + j\omega_{d_i} t}\pi(t-\tau_i), \tag{9}$$

where N, $\{\tau_i\}_{i=1}^{N}$, is a realization of the Poisson process (e.g., known).

Then, the energy received over $[0, \overline{\tau}_m]$ at some $t_0 \in [0, T_s]$ is defined by (see [2, pages 147–150]) $y_{\ell,N}(t_0) \triangleq (1/\overline{\tau}_m)\int_0^{\overline{\tau}_m} y^\star_{\ell,N}(t) y_{\ell,N}(t)dt$, which is the time average of the second moment of $y_{\ell,N}(t)$ based on a single realization over the interval $[0, \overline{\tau}_m]$. Further, if the multipath components are assumed to be resolved by the probing signal $\pi(t)$ (e.g., $|\tau_i - \tau_j| > T_c$, for all $i \neq j$), then

$$|y_{\ell,N}(t_0)|^2 = \frac{1}{\overline{\tau}_m}\sum_{i=1}^{N} r_i^2(t_0)\int_0^{\overline{\tau}_m}\pi^2(t-\tau_i)dt = \sum_{i=1}^{N} r_i^2(t_0). \tag{10}$$

The ensemble average power (due to a wideband signal transmission) is $\mathcal{E}_{\text{WB}} = \sum_{i=1}^{N} E[r_i^2(t_0)]$ $(= NE[r^2(t_0)]$ if r_i are i.i.d.). Our earlier equations calculate $\mathcal{E}_{\text{WB}}$ using ensemble average. In particular, $\mathcal{E}_{\text{WB}}$ corresponds to

$$\overline{y^2_{\ell,N}}(t) = \frac{1}{T_s}\sum_{i=1}^{N}\int_0^{T_s} E[r_i^2(\tau)]\pi^2(t-\tau)d\tau \approx \frac{\overline{\tau}_m}{T_s}\sum_{i=1}^{N} E[r_i^2(t)], \tag{11}$$

which is obtained under the assumption that $N(T_s) = N$ is fixed, $\lambda_T(t) = \lambda$ is a constant, and $Ey_\ell(t) = 0$. On the other hand, under the assumptions of Lemma 1, assuming constant rates $\lambda_T(t) = \lambda$ and $Ey_\ell(t) = 0$, we have from (7) that

$$\begin{aligned} E[|y_\ell(t)|^2] &= \lambda\int_0^{T_s} E[r^2(\tau)]\pi^2(t-\tau)d\tau \\ &= \lambda\overline{\tau}_m\sum_{i=1}^{N} E[r_i^2(t_0)] \\ \text{if } r(\tau) &= \sum_{i=1}^{N} r_i(t_0)\delta(\tau-t_0),\ t_0 \in [t-T_c, t], \end{aligned} \tag{12}$$

which is proportional to (10) and (11).

(4) *Narrowband transmission.* Consider next the transmission into the channel (9) of a continuous-wave signal, $x_\ell(t) = 1$. Then, the received power, given the realization of $\{N(t);\, 0 \leq t \leq T_s\}$, is

$$\begin{aligned} &\mathcal{P}_{\text{CW}} \\ &= E\left[\left|\sum_{i=1}^{N} r_i e^{j\phi_i} e^{-j(\omega_c+\omega_{d_i})\tau_i + j\omega_{d_i} t}\right|^2\right] \\ &= \sum_{i=1}^{N} E[r_i^2] + \sum_{\substack{i,m=1\\ i\neq m}}^{N} E\Big[r_i r_m e^{j(\phi_i-\phi_m)} e^{-j[\omega_c(\tau_i-\tau_m)-(\omega_{d_i}\tau_i-\omega_{d_m}\tau_m)]}\Big] \\ &\qquad \times e^{j(\omega_{d_i}-\omega_{d_m})t}. \end{aligned} \tag{13}$$

On the other hand, if $N(T_s) = N$ and $\lambda =$ constant, then by (4) letting $x_\ell(t) = 1$ yields

$$\begin{aligned} &\overline{y^2_{\ell,N}}(t) \\ &= \frac{1}{T_s}\int_0^{T_s}\sum_{i=1}^{N} E[r_i^2(\tau)]d\tau \\ &\quad + \frac{1}{T_s^2}\int_0^{T_s}\sum_{\substack{i,m=1\\ i\neq m}}^{N} E[r_i(\tau_i) r_m(\tau_m) e^{j(\phi_i-\phi_m)} \\ &\qquad \times e^{-j[\omega_c(\tau_i-\tau_m)-(\omega_{d_i}\tau_i-\omega_{d_m}\tau_m)]}] e^{j(\omega_{d_i}-\omega_{d_m})t} d\tau_i\, d\tau_m, \end{aligned} \tag{14}$$

which is proportional to (13). Clearly, the above comparisons indicate the consistency of the ensemble averages based on our model and analysis with respect to the analysis found in [2], even for the simple homogeneous Poisson process.

Correlation and covariance

The correlation of $y_\ell(t_1)$ and $y_\ell(t_2)$ is $R_{y_\ell}(t_1, t_2) \triangleq E[y^\star_\ell(t_1) y_\ell(t_2)] = E[\sum_{i=1}^{N(T_s)} h^\star_\ell(t_1, \tau_i; \mathbf{m}_i(\tau_i)) \sum_{i=1}^{N(T_s)} h_\ell(t_2, \tau_i; \mathbf{m}_i(\tau_i))]$, and the covariance is

$$\begin{aligned} &C_{y_\ell}(t_1, t_2) \\ &\triangleq R_{y_\ell}(t_1, t_2) - E[y^\star_\ell(t_1)]E[y_\ell(t_2)] \\ &= \sum_{k=1}^{\infty} R_{y_{\ell,k}}(t_1, t_2)\text{Prob}(N(T_s) = k) - E[y^\star_\ell(t_1)]E[y_\ell(t_2)], \end{aligned} \tag{15}$$

where

$$\begin{aligned} &R_{y_{\ell,k}}(t_1, t_2) \\ &\triangleq E[y^\star_{\ell,k}(t_1) y_{\ell,k}(t_2)] \\ &= E\left[\sum_{i=1}^{k} h^\star_\ell(t_1, \tau_i; \mathbf{m}_i(\tau_i)) h_\ell(t_2, \tau_i; \mathbf{m}_i(\tau_i))\right] \\ &\quad + E\left[\sum_{\substack{i,j=1\\ i\neq j}}^{k} h^\star_\ell(t_1, \tau_i; \mathbf{m}_i(\tau_i)) h_\ell(t_2, \tau_j; \mathbf{m}_j(\tau_j))\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \frac{1}{\int_0^{T_s} \lambda_T(t)dt} \int_0^{T_s} \lambda_T(\tau) E\left[r_i^2(\tau) e^{j\omega_{d_i}(t_2-t_1)}\right] \\
&\quad \times x_\ell^\star(t_1-\tau) x_\ell(t_2-\tau) d\tau \\
&\quad + \sum_{\substack{i,m=1\\ i\neq m}}^{k} \frac{1}{\int_0^{T_s} \lambda_T(t)dt} E\Bigg[e^{-j(\phi_i-\phi_m)} e^{-j(\omega_{d_i}t_1-\omega_{d_m}t_2)} \\
&\quad \times \int_0^{T_s} \lambda_T(\tau) r_i(\tau) e^{j(\omega_c+\omega_{d_i})\tau} x_\ell^\star(t_1-\tau) d\tau \\
&\quad \times \frac{1}{\int_0^{T_s} \lambda_T(t)dt} \int_0^{T_s} \lambda_T(\tau) r_m(\tau) e^{-j(\omega_c+\omega_{d_m})\tau} x_\ell^\star(t_2-\tau) d\tau \Bigg].
\end{aligned} \tag{16}
$$

The above expression is further simplified by invoking Assumption 1.

Lemma 2. *Consider model* (2)-(3) *under Assumption 1. Then,*

$$
\begin{aligned}
&R_{y_\ell}(t_1,t_2) \\
&\quad = \lambda \int_0^{T_s} \lambda_c(\tau) E\left[h_\ell^\star(t_1,\tau;\mathbf{m}(\tau)) h_\ell(t_2,\tau;\mathbf{m}(\tau))\right] d\tau \\
&\qquad + \lambda \int_0^{T_s} \lambda_c(\tau) E\left[h_\ell^\star(t_1,\tau;\mathbf{m}(\tau))\right] d\tau \\
&\qquad \times \lambda \int_0^{T_s} \lambda_c(\tau) E\left[h_\ell(t_2,\tau;\mathbf{m}(\tau))\right] d\tau \\
&\quad = \lambda \int_0^{T_s} \lambda_c(\tau) E\left[r^2(\tau) e^{j\omega_d(t_2-t_1)}\right] x_\ell^\star(t_1-\tau) x_\ell(t_2-\tau) d\tau \\
&\qquad + \lambda \int_0^{T_s} \lambda_c(\tau) e^{j\omega_c\tau} E\left[r(\tau) e^{-j\phi} e^{-j\omega_d(t_1-\tau)}\right] x_\ell^\star(t_1-\tau) d\tau \\
&\qquad \times \lambda \int_0^{T_s} \lambda_c(\tau) e^{-j\omega_c\tau} E\left[r(\tau) e^{j\phi} e^{j\omega_d(t_2-\tau)}\right] x_\ell(t_2-\tau) d\tau, \\
&\qquad\qquad 0 \le t_1,\ t_2 \le T_s,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&C_{y_\ell}(t_1,t_2) \\
&\quad = \lambda \int_0^{T_s} \lambda_c(\tau) E\left[h_\ell^\star(t_1,\tau;\mathbf{m}(\tau)) h_\ell(t_2,\tau;\mathbf{m}(\tau))\right] d\tau \\
&\quad = \lambda \int_0^{T_s} \lambda_c(\tau) E\left[r^2(\tau) e^{j\omega_d(t_2-t_1)}\right] x_\ell^\star(t_1-\tau) x_\ell(t_2-\tau) d\tau, \\
&\qquad\qquad 0 \le t_1,\ t_2 \le T_s.
\end{aligned} \tag{18}
$$

Proof. Follow the derivation of Lemma 1. □

Remark 2. Next we illustrate how the rate of Poisson process affects both the Doppler power spectrum and the power delay profile. Consider the results of Lemma 2 when $t_1 = t$, $t_2 = t + \Delta t$, and $x_\ell(t) = 1$, for all $t \in [0, T]$ (e.g., a narrowband signal), and for fixed $\tau_i = \tau$, $\omega_{d_i}(\tau) = (2\pi\nu(\tau)/\lambda_\omega)\cos\theta_i$, where $\nu(\tau_i)$ is the speed of the mobile, corresponding to the ith path, λ_ω is the wavelength, and θ_i is uniformly distributed in $[0, 2\pi]$ [4, 5] (the dependence of ω_{d_i} on τ is obviously incorporated in the previous results). We will compute the autocorrelation, Doppler spread, and power delay profile of the channel.

(1) *Doppler power spectrum.* Under the above assumptions (and assuming $Ey_\ell(t) = 0$), the autocorrelation of $y_\ell(t)$ is

$$
R_{y_\ell}(\Delta t) = \lambda \int_0^{T_s} \lambda_c(\tau) E\left[r^2(\tau) e^{j\omega_d(\tau)\Delta t}\right] d\tau, \tag{19}
$$

and its power spectral density is

$$
\mathcal{F}_{\Delta t}\{R_{y_\ell}(\Delta t)\} = \lambda \int_0^{\infty} \int_0^{T_s} \lambda_c(\tau) E\left[r^2(\tau) e^{j\omega_d(\tau)\Delta t}\right] e^{-j2\pi f\Delta t} d\tau\, dt. \tag{20}
$$

Moreover, if $r(\tau)$ and $\omega_d(\tau)$ are independent (as commonly assumed) and $\lambda_c(t) = \sum_{i=1}^{N} \delta(t - t_i)$, then $R_{y_\ell}(\Delta t) = \lambda\sum_{i=1}^{N} E[r^2(t_i)] \times J_0((2\pi\nu(t_i)/\lambda)\Delta t)$, which is a commonly known expression, where $J_0(\cdot)$ is a Bessel function of first kind of zero order (see [5] for $N = 1$), and $\mathcal{F}_{\Delta t}\{C_{y_\ell}(\Delta t)\} = \lambda\sum_{i=1}^{N} E[r^2(t_i)] \times S_{D_i}(f)$, where

$$
S_{D_i}(f) = \begin{cases} \dfrac{1}{2\pi} \dfrac{\lambda_\omega}{\nu(t_i)} \dfrac{1}{\sqrt{1-\left(f\lambda_\omega/\nu(t_i)\right)^2}}, & |f| \le \dfrac{\nu(t_i)}{\lambda_\omega}, \\ 0, & \text{elsewhere}, \end{cases} \tag{21}
$$

for $1 \le i \le N$. Thus, $S_{D_i}(f)$ is the Doppler spread predicted in [4, 5] for a two-dimensional propagation model. More general models such as those found in [5] can be considered as well.

(2) *Power delay profile.* Under the above assumptions (and assuming $Ey_\ell(t) = 0$), the power delay profile of $y_\ell(t)$, denoted by $\phi(\tau)$, is obtained from (17) by letting $t_1 = t_2 = t$ and letting $x(t)$ be a delta function, which implies that $\phi(\tau) = \lambda_T(\tau)E[r^2(\tau)]$. Clearly, the rate of the Poisson process determines the shape of the power delay profile as expected. Note that in practise one can obtain the rate $\lambda_T(\cdot)$ via maximum-likelihood methods by noisy channel measurements.

However, if $r(\tau)$ and $\omega_d(\tau)$ are not independent, then more general expressions for the autocorrelation and Doppler spread are obtained.

3. POWER SPECTRAL DENSITIES

Throughout this section, it is assumed (for simplicity) that $\{r_i(\tau_i)\}_{i\ge1}$ are independent of $\tau_i's$, and thus we denote them by $\{r_i\}_{i\ge1}$; $N(T_s)$ is homogeneous Poisson. However, if one considers the τ-dependent attenuations $\{r_i(\tau)\}_{i\ge1}$, then as a function of τ, each $r_i : \Omega \times [0, T_s] \to [0, \infty)$, and therefore each r_i is a random process as a function of τ. In this case, the results will also hold provided that one assumes that $\{r_i(\tau)\}_{\tau \geq 0}$ as functions of τ are wide-sense stationary (because $E[r^2(\tau)]$ and $E[r(\tau)]$ are independent of τ).

Power spectral density

The expressions for the correlation function and the covariance function (assuming that $\int_{t-T_s}^{t}$ is denoted by $\int_{\infty}^{\infty}$) are

$$C_{y_\ell}(\tau) = \lambda E[r^2 e^{j\omega_d \tau}] \int_{-\infty}^{\infty} x_\ell^\star(\alpha) x_\ell(\tau + \alpha) d\alpha, \tag{22}$$

$$\begin{aligned} R_{y_\ell}(\tau) = {} & C_{y_\ell}(\tau) + \lambda E\Big[re^{-j\phi} \int_{-\infty}^{\infty} e^{-j(\omega_c+\omega_d)\alpha} x_\ell^\star(\alpha) d\alpha\Big] \\ & \times \lambda E\Big[re^{j\phi} \int_{-\infty}^{\infty} e^{j(\omega_c+\omega_d)\alpha} e^{j\omega_d \tau} x_\ell(\tau+\alpha) d\alpha\Big]. \end{aligned} \tag{23}$$

Taking Fourier transforms, we obtain the following result.

Theorem 1. *Consider model* (2)-(3) *under Assumption 1 with $\{r_i(\tau)\}_{i\geq 1}$ being independent of τ, and consider $N(t)$ a homogeneous Poisson process with rate $\lambda \geq 0$. Define the centered processes $y_{\ell,c}(t) \triangleq y_\ell(t) - \overline{y_\ell}(t)$, $y_c(t) \triangleq y(t) - \overline{y}(t)$, and*

$$\begin{aligned} X_\ell(j\omega) &\triangleq \int_{-\infty}^{\infty} x_\ell(t) e^{-j\omega t} dt, \\ X(j\omega) &\triangleq \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \\ x(t) &= \operatorname{Re}\{x_\ell(t) e^{j(\omega_c+\omega_d)t + j\phi}\}. \end{aligned} \tag{24}$$

The power spectral densities of the centered processes $y_{\ell,c}(t)$ and $y_c(t)$ are

$$S_{y_{\ell,c}}(j\omega) \triangleq \mathcal{F}_\tau\{C_{y_\ell}(\tau)\} = \lambda E[r^2 |X_\ell(j(\omega - \omega_d))|^2], \tag{25}$$

$$S_{y_c}(j\omega) \triangleq \mathcal{F}_\tau\{R_{y_c}(\tau)\} = \lambda E[r^2\, |X(j\omega)|^2], \tag{26}$$

and the power spectral densities of $y_\ell(t)$ and $y(t)$ are

$$\begin{aligned} S_{y_\ell}(j\omega) &\triangleq \mathcal{F}_\tau\{R_{y_\ell}(\tau)\} \\ &= S_{y_\ell,c}(j\omega) + 2\pi\lambda^2 E[re^{j\phi} X_\ell(-j(\omega_c + \omega_d))] \\ &\quad \times E[re^{-j\phi} X_\ell^\star(-j(\omega_c+\omega_d))]\delta(w + \omega_c), \end{aligned} \tag{27}$$

$$\begin{aligned} S_y(j\omega) &\triangleq \mathcal{F}_\tau\{R_y(\tau)\} \\ &= S_{y_c}(j\omega) + 2\pi\lambda^2 (E[rX(0)])^2 \delta(\omega). \end{aligned} \tag{28}$$

Further, assuming $\gamma_1(t) = \lambda \int_0^{T_s} E[h(t,\tau;\mathbf{m})]d\tau = 0$, the power spectral density of $y^2(t)$ is

$$\begin{aligned} S_{y^2}(j\omega) &\triangleq \mathcal{F}_\tau\{C_{y^2}(\tau)\} \\ &= \frac{\lambda^2}{\pi} E[r^2 |X(j\omega)|^2] * E[r^2 |X(j\omega)|^2] \\ &\quad + 2\pi\lambda^2 \overline{E}^2 \delta(\omega) + \frac{\lambda}{4\pi^2} E[|r^2 X(j\omega) * X(j\omega)|^2], \end{aligned} \tag{29}$$

where $\overline{E} = E[\int_{-\infty}^{\infty} r^2 x^2(t) dt]$.

Remark 3. The behavior of the power spectral densities for high and low rates λ is obtained as follows.

(1) *High-rate approximation.* If λ is sufficiently large, then the third term in (29) can be neglected and the power spectrum of $y^2(t)$ consists of only the first and second right-hand side terms of (29).

(2) *Low-rate approximation.* If λ is small, then the probability that the terms $h(t - \tau_i; \mathbf{m}_i)$ and $h(t - \tau_j; \mathbf{m}_j)$ have significant overlaps, for $i \neq j$, is very small, hence the approximation $y^2(t) = \sum_{i=1}^{N(T_s)} h^2(t - \tau_i; \mathbf{m}_i)$. This is equivalent to assuming that the paths do not overlap. As described earlier, the power spectral density expressions are important in receiver designing and for modeling the interference.

4. DISTRIBUTIONS AND MOMENT-GENERATING FUNCTIONS

Let $I_{\{A\}}$ denote the indicator function ofthe event A, which is 1 if the event A occurs and zero otherwise. The probability density function and moment-generating functions of $y(t)$ and $y_\ell(t)$ are, respectively, defined by $f_y(x,t)dx \triangleq E[I_{\{y(t)\in dx\}}]$, $f_{y_\ell}(x_\ell, t)dx_\ell \triangleq E[I_{\{y_\ell(t)\in dx_\ell\}}]$, $\Phi_y(s,t) \triangleq E[e^{sy(t)}]$, $\Phi_{y_\ell}(\rho, t) \triangleq E[e^{j\operatorname{Re}(\rho^\star y_\ell(t))}]$, $s \triangleq j\omega$, $\rho \in \mathcal{C}$. Consider the real signal $y(t)$; for fixed $N(T_s) = k$, the density of $y(t)$ is $f_{y_k}(x,t)dx \triangleq E[I_{\{y(t)\in dx\}} \mid N(T_s) = k] = \text{Prob}\{\sum_{i=1}^{k} h(t,\tau_i;\mathbf{m}_i(\tau_i)) \in dx\}$. Assuming a homogeneous Poisson process (for simplicity of presentation), we obtain $f_y(x,t) = e^{-\lambda T_s} \sum_{k=1}^{\infty} f_{y_k}(x,t)((\lambda T_s)^k/k!)$. For fixed $N(T_s) = k$, the moment-generating function of $y(t)$ is

$$\begin{aligned} &\Phi_{y_k}(s,t) \\ &\quad \triangleq E\Big[\exp\Big\{s \sum_{i=1}^{N(T_s)} h(t,\tau_i;\mathbf{m}_i(\tau_i))\Big\} \mid N(T_s) = k\Big] \end{aligned} \tag{30}$$

$$\begin{aligned} &\quad = \frac{1}{T_s^k} \int_0^{T_s} d\tau_1 \int_0^{T_s} d\tau_2 \cdots \int_0^{T_s} d\tau_k E\Big[\prod_{i=1}^{k} e^{sh(t,\tau_i;\mathbf{m}_i(\tau_i))}\Big] \\ &= \prod_{i=1}^{k} \frac{1}{T_s} \int_0^{T_s} d\tau E[e^{sh(t,\tau;\mathbf{m}_i(\tau))}] \\ &\qquad \text{if } \{e^{sh(t,\tau_i;\mathbf{m}_i(\tau_i))}\}_{i\geq 1} \text{ are uncorrelated.} \end{aligned} \tag{31}$$

Clearly, the above calculations hold for the low-pass equivalent complex representation as well, leading to the following results. The above expressions are simplified further by invoking Assumption 1.

Theorem 2. *Consider model* (2)-(3) *and Assumption 1.*

(1) *The characteristic function of* $y(t)$ *is*

$$\begin{aligned}\Phi_y(s,t) &\triangleq E[e^{sy(t)}]\\ &= \exp\left\{\lambda\int_0^{T_s}\lambda_c(\tau)E[e^{sh(t,\tau;\mathbf{m}(\tau))}-1]d\tau\right\},\quad s\triangleq j\omega,\end{aligned} \tag{32}$$

and its density is

$$\begin{aligned}f_y(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{-j\omega x}\\ &\quad\times\exp\left\{\lambda\int_0^{T_s}\lambda_c(\tau)E[e^{sh(t,\tau;\mathbf{m}(\tau))}-1]d\tau\right\}.\end{aligned} \tag{33}$$

Moreover,

$$\begin{aligned}\Psi_y(j\omega,t) &\triangleq \ln E[e^{sy(t)}]\\ &= \sum_{k=1}^{\infty}\frac{(j\omega)^k}{k!}\gamma_k(t)\quad\text{provided that } \gamma_k(t)<\infty,\end{aligned} \tag{34}$$

where

$$\gamma_k(t) = \lambda\int_0^{T_s}\lambda_c(\tau)E[h(t,\tau;\mathbf{m}(\tau))]^k d\tau, \tag{35}$$

is the kth cumulant of $y(t)$, *and* $\gamma_1(t)=E[y(t)]$ *and* $\gamma_2(t)=\mathrm{Var}(y(t))$.

(2) *The characteristic function of* $y_\ell(t)$ *is*

$$\begin{aligned}\Phi_{y_\ell}(\rho,t) &\triangleq E[e^{j\mathrm{Re}(\rho^\star y_\ell(t))}]\\ &= \exp\left\{\lambda\int_0^{T_s}\lambda_c(\tau)E[e^{j\mathrm{Re}(\rho^\star h_\ell(t,\tau;\mathbf{m}(\tau)))}-1]d\tau\right\},\end{aligned} \tag{36}$$

where $\rho\triangleq\rho_R+j\rho_I$, *and its density is*

$$\begin{aligned}f_{y_\ell}(x_\ell,t) &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}d\rho_R\,d\rho_I\,e^{-j\mathrm{Re}(\rho^\star x_\ell)}\\ &\quad\times\exp\left\{\lambda\int_0^{T_s}\lambda_c(\tau)E[e^{j\mathrm{Re}(\rho^\star h(t,\tau;\mathbf{m}(\tau)))}-1]d\tau\right\}.\end{aligned} \tag{37}$$

Moreover, for $m,n>0$ *integers*

$$\begin{aligned}&E[(y_\ell^\star(t))^k(y_\ell(t))^m]\\ &\quad= (-2j)^{k+m}\left(\frac{\partial}{\partial\rho}\right)^k\left(\frac{\partial}{\partial\rho^\star}\right)^m\Phi_{y_\ell}(\rho,t)\big|_{\rho=0},\end{aligned} \tag{38}$$

$$\Psi_{y_\ell}(\rho,t)\triangleq\ln E[e^{j\mathrm{Re}(\rho^\star y_\ell(t))}] = \sum_{k=1}^{\infty}j^k\frac{\gamma_{\ell,k}(t)}{k!}, \tag{39}$$

where

$$\begin{aligned}\gamma_{\ell,k}(t) &= \lambda\int_0^{T_s}\lambda_c(\tau)E[\mathrm{Re}(\rho^\star h_\ell(t,\tau;\mathbf{m}(\tau)))]^k d\tau,\\ E[y_\ell(t)] &= (-2j)j\frac{\partial}{\partial\rho^\star}\gamma_{\ell,1}(t),\\ E[y_\ell^\star(t)] &= (-2j)j\frac{\partial}{\partial\rho}\gamma_{\ell,1}(t),\\ \mathrm{Var}(y(t)) &= (-2j)^2j^2\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho^\star}\frac{\gamma_{\ell,2}(t)}{2!}.\end{aligned} \tag{40}$$

Proof. The derivation is similar to that found in [13, page 156-157]. □

The above theorem gives closed-form expressions for all the moments of $y(t)$ and $y_\ell(t)$ and their real and imaginary parts. These expressions are easily computed for the example of Remark 2.

Central limit theorem

The joint characteristic functions of $y(t_1),\ldots,y(t_n)$ and $y_\ell(t_1),\ldots,y_\ell(t_n)$ along with their cumulants are obtained following the derivation of Theorem 2.

Corollary 1. *Consider model* (2)-(3) *under Assumption 1.*

(1) *The joint characteristicfunction of* $y(t_1),\ldots,y(t_n)$ *is*

$$\begin{aligned}&\Phi_{\mathbf{y}}(s_1,t_1;\ldots;s_n,t_n)\\ &\quad\triangleq E\left\{\exp\left(\sum_{i=1}^n s_iy(t_i)\right)\right\},\\ &\quad= \exp\left\{\lambda\int_0^{T_s}\lambda_c(\tau)E\left[\exp\left(\sum_{i=1}^n s_ih(t_i,\tau;\mathbf{m}(\tau))\right)-1\right]d\tau\right\},\\ &\qquad\qquad \mathbf{y}(t) = (y(t_1),y(t_2),\ldots,y(t_n))'\in\mathfrak{R}^n,\end{aligned} \tag{41}$$

where $s_i\triangleq j\omega_i$, $1\le i\le n$.

(2) *The joint characteristic function of* $y_\ell(t_1),\ldots,y_\ell(t_n)$ *is*

$$\begin{aligned}&\Phi_{\mathbf{y}_\ell}(\rho_1,t_1;\ldots;\rho_n,t_n)\\ &\triangleq E[\exp\{j\mathrm{Re}(\rho^\dagger\mathbf{y}_\ell(t))\}]\\ &= \exp\left\{\lambda\int_0^{T_s}\lambda_c(\tau)E\left[\exp\left(j\sum_{i=1}^n[\rho_{R_i}\mathrm{Re}(h_\ell(t_i,\tau;\mathbf{m}(\tau)))\right.\right.\right.\\ &\qquad\left.\left.\left.+\rho_{I_i}\mathrm{Im}(h_\ell(t_i,\tau;\mathbf{m}(\tau)))]\right)-1\right]d\tau\right\}\\ &= \exp\left\{\lambda\int_0^{T_s}\lambda_c(\tau)E[\exp(j\mathrm{Re}(\rho^\dagger\mathbf{h}_\ell(t,\tau;\mathbf{m}(\tau))))-1]d\tau\right\},\\ &\qquad\qquad \mathbf{y}_\ell(t) = (y_\ell(t_1),\ldots,y_\ell(t_n))'\in\mathfrak{C}^n,\end{aligned} \tag{42}$$

where $\mathbf{h}_\ell(t,\tau;\mathbf{m}(\tau)) = (h_\ell(t_1,\tau;\mathbf{m}(\tau)),\ldots,h_\ell(t_n,\tau;\mathbf{m}(\tau)))'$.

The joint moment-generating function of the complex random variables $y_\ell(t_1),\ldots,y_\ell(t_n)$ is

$$E\left[\prod_{i=1}^{n}(y_\ell^\star(t_i))^{k_i}\prod_{i=1}^{n}(y_\ell(t_i))^{m_i}\right] = (-2j)^{\sum_{i=1}^{n}(k_i+m_i)}\prod_{i=1}^{n}\left(\frac{\partial}{\partial\rho_i}\right)^{k_i}\prod_{i=1}^{n}\left(\frac{\partial}{\partial\rho_i^\star}\right)^{m_i} \times\Phi_{\mathbf{y}_\ell}(\rho_1,t_1,\ldots;\rho_n,t_n)\,|_{\rho=0}. \tag{43}$$

Corollary 1 gives closed-form expressions for joint statistics of $\{y_\ell(t)\}_{t\geq 0}$ and $\{y(t)\}_{t\geq 0}$, including correlations and higher-order statistics. These are easily computed for the example of Remark 2.

We will show next that for large λ, compared to the time constants of the signal x, the joint distribution of $y(t_1),\ldots,y(t_n)$ is normal, thus establishing a central limit theorem for $\{y(t)\}_{t\geq 0}$ as a random process. Further, we will illustrate that similar results hold for the complex random variables $y_\ell(t_1),\ldots,y_\ell(t_n)$. This is a generalization of the Gaussianity of shot noise described by Rice in [7, 8].

To this end, define the centered random variables $y_c(t_i)\triangleq(y(t_i)-\overline{y}(t_i))/\sigma_y(t_i)$ and $\sigma_y(t_i)=\sqrt{\mathrm{Var}(y(t_i))}$, $1\leq i\leq n$. According to Corollary 1, the joint characteristic function of the centered random variables $y_c(t_1),y_c(t_2),\ldots,y_c(t_n)$ is

$$\begin{aligned}&\Phi_{\mathbf{y}_c}(j\omega_1,t_1;\ldots,j\omega_n,t_n)\\ &\triangleq E\left[\exp\left\{\sum_{i=1}^{n}s_iy_c(t_i)\right\}\right]\Bigg|_{s_i=j\omega_i}\\ &=\exp\left\{-j\sum_{i=1}^{n}\omega_i\frac{\overline{y}(t_i)}{\sigma_y(t_i)}\right\}\\ &\quad\times\exp\Bigg\{\lambda\int_0^{T_s}\lambda_c(\tau)E\\ &\qquad\times\left[\exp\left(\sum_{i=1}^{n}\frac{j\omega_i}{\sigma_y(t_i)}h(t_i,\mathbf{m}(\tau);\tau)\right)-1\right]d\tau\Bigg\}.\end{aligned} \tag{44}$$

Expand in power series (assuming an absolute convergent series with finite integrals):

$$\begin{aligned}&\lambda\int_0^{T_s}\lambda_c(\tau)E\left[\exp\left\{\sum_{i=1}^{n}j\frac{\omega_i}{\sigma_y(t_i)}h(t_i,\tau;\mathbf{m}(\tau))\right\}-1\right]d\tau\\ &=\lambda\int_0^{T_s}\lambda_c(\tau)E\left[\sum_{i=1}^{n}\frac{j\omega_i}{\sigma_y(t_i)}h(t_i,\tau;\mathbf{m}(\tau))\right]d\tau\\ &\quad+\frac{1}{2}\lambda\int_0^{T_s}\lambda_c(\tau)E\left[\sum_{i=1}^{n}\frac{j\omega_i}{\sigma_y(t_i)}h(t_i,\tau;\mathbf{m}(\tau))\right]^2d\tau+\cdots.\end{aligned} \tag{45}$$

Since $\sigma_y(t_i)$ is proportional to $\lambda^{1/2}$, the first term in the power series expansion is of order $\lambda^{1/2}$, the second term is of order 1, the third term is of order $1/\lambda^{1/2}$, and the kth is term of order $\lambda/\lambda^{k/2}=\lambda^{-(k-2)/2}$. Hence, for large λ, we have the following approximation (neglecting terms of order $\lambda^{-(k-2)/2}$, $k\geq 3$):

$$\begin{aligned}&\lambda\int_0^{T_s}\lambda_c(\tau)E\left[\exp\left(\sum_{i=1}^{n}j\omega_ih(t_i,\tau;\mathbf{m}(\tau))\right)-1\right]d\tau\\ &\quad\approx\lambda\int_0^{T_s}\lambda_c(\tau)E\left[\sum_{i=1}^{n}\frac{j\omega_i}{\sigma_y(t_i)}h_\ell(t_i,\tau;\mathbf{m}(\tau))\right]d\tau\\ &\quad+\frac{1}{2}\lambda\int_0^{T_s}\lambda_c(\tau)E\left[\sum_{i=1}^{n}\frac{j\omega_i}{\sigma_y(t_i)}h(t_i,\tau;\mathbf{m}(\tau))\right]^2d\tau.\end{aligned} \tag{46}$$

Substituting (46) into (44), the first right-hand side term in (44) is cancelled, hence

$$\begin{aligned}&\Phi_{\mathbf{y}_c}(j\omega_1,t_1;\ldots;j\omega_n,t_n)\\ &\quad\approx\exp\left\{-\frac{\lambda}{2}\int_0^{T_s}\lambda_c(\tau)E\left[\sum_{i=1}^{n}\frac{\omega_i}{\sigma_y(t_i)}h(t_i,\tau;\mathbf{m}(\tau))\right]^2d\tau\right\}.\end{aligned} \tag{47}$$

The last expression shows that the joint characteristic function is quadratic in $\{\omega_j\}_{j=1}^{n}$. Hence, $y_c(t_1),\ldots,y_c(t_n)$ are approximately Gaussian, with zero mean and the covariance matrix identified. Moreover, $y_c(t_j)\sim N(0;1)$, $1\leq j\leq n$. In the limit, as $\lambda\to\infty$, the above approximation becomes exact. In general, the above central limit result holds as certain parameters entering $h(\cdot,\cdot;\cdot)$ approach their limits, other than $\lambda\to\infty$. If we consider the example of Remark 2, and let $\lambda_T(t)$ be a constant (say λ), then the Gaussianity statement holds provided that $T_s\to\infty$ (this is consistent with the understanding that as T_s becomes large, more paths are present and hence the central limit theorem will hold).

Lemma 3. *Consider model* (2)-(3) *under Assumption 1.*

(1) *The joint characteristic function of the centered random variables*

$$y_c(t_i)\triangleq\frac{y(t_i)-\overline{y}(t_i)}{\sigma_y(t_i)},\quad \sigma_y(t_i)=\sqrt{\mathrm{Var}(y(t_i))},\qquad 1\leq i\leq n, \tag{48}$$

is in the limit, as $\lambda\to\infty$, and is Gaussian with

$$\begin{aligned}&\lim_{\lambda\to\infty}\Phi_{\mathbf{y}_c}(j\omega_1,t_1;\ldots;j\omega_n,t_n)\\ &\quad\triangleq\lim_{\lambda\to\infty}E\left[\exp\left\{\sum_{i=1}^{n}s_iy_c(t_i)\right\}\right],\\ &\quad=\exp\left\{-\frac{\lambda}{2}\int_0^{T_s}\lambda_c(\tau)E\left[\sum_{i=1}^{n}\frac{\omega_i}{\sigma_y(t_i)}h(t_i,\tau;\mathbf{m}(\tau))\right]^2d\tau\right\},\\ &\qquad s_i\triangleq j\omega_i\ (1\leq i\leq n).\end{aligned} \tag{49}$$

(2) *The joint characteristic function of the centered random variables*

$$y_{\ell,c}(t_i)\triangleq\frac{y_\ell(t_i)-\overline{y_\ell}(t_i)}{\sigma_{y_\ell}(t_i)},\quad \sigma_{y_\ell}(t_i)=\sqrt{\mathrm{Var}(y_\ell(t_i))},\qquad 1\leq i\leq n, \tag{50}$$

is in the limit, as $\lambda \to \infty$, and is complex Gaussian with

$$\lim_{\lambda\to\infty} \Phi_{\mathbf{y}_{\ell,c}}(\rho_1, t_1;\ldots;\rho_n, t_n) \triangleq \lim_{\lambda\to\infty} E[\exp\{j\mathrm{Re}(\rho^{\dagger}\mathbf{y}_{\ell,c}(t))\}] = \exp\Bigg\{-\frac{\lambda}{2}\int_0^{T_s}\lambda_c(\tau)E \times\sum_{i=1}^{n}\bigg[\frac{\rho_{R_i}}{\sigma_{y_\ell}(t_i)}\mathrm{Re}(h_\ell(t_i,\tau,\mathbf{m}(\tau))) + \frac{\rho_{I_i}}{\sigma_{y_\ell}(t_i)}\mathrm{Im}(h_\ell(t_i,\tau,\mathbf{m}(\tau)))\bigg]^2 d\tau\Bigg\}, \tag{51}$$

where $\mathbf{y}_{\ell,c}(t) = (y_{\ell,c}(t_1),\ldots,y_{\ell,c}(t_n))' \in \mathbb{C}^n$.

Proof. (1) The proof follows from the above construction. (2) Equation (51) is obtained by following exactly the same procedure as in (1) (see also [13, page 157]). □

Remark 4. Next, we discuss the implications of the previous lemma and some generalizations of the results obtained.

(1) Clearly, in (49) and (51), the exponents are quadratic functions of $\{\omega_i\}_{i=1}^n$ and $\{\rho_{R_i},\rho_{I_i}\}_{i=1}^n$, respectively; therefore one can easily specify the correlation properties of the received Gaussian signal, irrespective of the transmitted input signal. Unlike [5] in which Gaussianity of the inphase and quadrature components is derived, the last theorem shows Gaussianity of the received signal which is multipath, and identifies one of the parameters which is responsible for such Gaussianity to hold. Further, in many places it is often conjectured that for a large number of paths the inphase and quadrature components of the received signal are Gaussian. Some authors argue that the low-pass representation of the band-limited channel impulse response is complex Gaussian. Lemma 3 establishes the above conjecture in the limit as the rate of the Poisson process tends to infinity, by identifying the mean and the covariance of the Gaussian process. Clearly, as λ increases the number of paths received in a given observation interval increases, which then implies that resolvability of the paths is highly unlikely. Note that Lemma 3 can be used to compute the second-order statistics of the inphase and quadrature components. The mean of the inphase component is $E[I(t)] = \lambda\int_0^{T_s}\lambda_c(\tau)E[r(\tau)\cos(\omega_c\tau+\omega_d(t-\tau))]d\tau$, and its covariance is $C_I(t_1,t_2) = \lambda\int_0^{T_s}\lambda_c(\tau)E[r^2(\tau)\cos(\omega_c\tau+\omega_d(t_1-\tau))\cos(\omega_c\tau + \omega_d(t_2-\tau))]d\tau$.

(2) Every result obtained also holds for random signals x and x_ℓ, such as CDMA signals, provided that the expectation operation $E[\cdot]$ operates on the signals x and x_ℓ as well. Moreover, if the counting process is neither orderly nor independent increment, then the rate of the counting process, namely, $\lambda\times\lambda_c(t)$, should be random. This will be the case if λ is a random variable, and the earlier results will hold provided that there is an additional expectation with respect to the distribution of the random variable λ. Finally, we point out that one may use the current paper and the methodology in [9] to derive expressions for interference signals.

5. CONCLUSION

This paper presents a unified framework for studying the statistical characteristics of multipath fading channels, which can be viewed as a generalization of the mathematical analysis of the shot-noise effect. These include the second-order statistics, power spectral densities, and central limit theorems which are generalizations of Campbell's theorem. In the case of nonhomogeneous Poisson process, each propagation environment is identified with the rate $\lambda_T(t) = \lambda\times\lambda_c(t)$, in which $\lambda_c(\cdot)$ acts as a filter in shaping the received signal. This rate is an important parameter which needs to be identified prior to any design considerations associated with wireless channels.

ACKNOWLEDGMENT

The research leading to this results has received funding from the Research Promotion Foundation of Cyprus under the grant $\phi\lambda HPO \setminus 0603 \setminus 06$, and from the European Community's Sixth Framework Program (FP6) under the Agreement no. IST-034413 and Project NET-ReFound.

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