Research Article

A Novel Method for Improving Fairness over Multiaccess Channels

Seyed Alireza Razavi and Ciprian Doru Giurcăneanu

Department of Signal Processing, Tampere University of Technology, P.O. Box 553, 33101 Tampere, Finland

Correspondence should be addressed to Seyed Alireza Razavi, alireza.razavi@tut.fi

It is known that the orthogonal multiple access (OMA) guarantees for homogeneous networks, where all users have almost the same received power, a higher degree of fairness (in rate) than that provided by successive interference cancellation (SIC). The situation changes in heterogeneous networks, where the received powers are very disparate, and SIC becomes superior to OMA. In this paper, we propose to partition the network into (almost) homogeneous subnetworks such that the users within each subnetwork employ OMA, and SIC is utilized across subnetworks. The newly proposed scheme is equivalent to partition the users into ordered groups. The main contribution is a practical algorithm for finding the ordered partition that maximizes the minimum rate. We also give a geometrical interpretation for the rate-vector yield by our algorithm. Experimental results show that the proposed strategy leads to a good tradeoff between fairness and the asymptotic multiuser efficiency.

1. Introduction and Preliminaries

Rate allocation in multiuser communication systems is an important task which should consider simultaneously the fairness and the spectral efficiency. This paper is focused on fairness of multiple-access (MA) schemes working under maximum spectral efficiency evaluated in terms of sum rate. The state of art is the method recently introduced in [1]. However, the main drawback of this algorithm is a significant decrease of the asymptotic multiuser efficiency (AME) [2–4]. We propose a new strategy that combines the strengths of two different MA schemes such that to guarantee a good tradeoff between fairness and AME.

1.1. System Model. Consider a single-antenna Gaussian MA channel with K users transmitting to the base station (BS). The system model can be written as [1, Example 1]

$$y = \sum_{k=1}^{K} h_k s_k + \omega, \qquad (1)$$

where y is the received signal, h_k models the fading channel from the kth user to the BS, and s_k is the symbol transmitted by the kth user. The additive noise ω is assumed to be white circular Gaussian with variance $N_0/2$ for each real and imaginary component. Under the hypothesis that the transmitting powers of the users are constrained such that $E[|s_k|^2] \leq p_k, k \in \{1, \ldots, K\}$, we have

$$\sum_{k=1}^{K} R_k \leq C_{\text{sum}}, \qquad (2)$$

$$C_{\text{sum}} = \log\left(1 + \frac{\sum_{k=1}^{K} P_k}{N_0}\right) \text{ bits/s/Hz}, \qquad (3)$$

where R_k is the rate of the kth user and $P_k = p_k|h_k|^2$. The interested reader can find in [5, Chapter 6] a comprehensive discussion on the significance of (2) and (3). The following two methods can be applied to achieve equality in (2):

(1) OMA: orthogonal multiple-access with degrees of freedom (DOF) allocated proportional to users' received powers;

(2) SIC: successive interference cancellation.

We refer to [5, Chapter 6] for more details on OMA, SIC, and the definition of DOF.

It is also pointed out in [5] that, whenever the received power is almost the same for all users, that is, the network is homogeneous, OMA guarantees a higher degree of fairness (in rate) than that provided by SIC. The situation changes in heterogeneous networks, where the received powers are very disparate: if the decoding is performed in the decreasing order of the received powers, then SIC becomes superior to OMA. However, the SIC systems have drawbacks which do not exist for OMA. Because the signals received from the users are estimated and subtracted from the composite signal one after the other, the inaccurate estimation for the current user makes the next users decoded unreliably. This deficiency becomes more severe when the number of users increases. In fact, it is known that SIC works well only when a specific disparity of the powers is enforced (see, e.g., [6, 7] and Chapter 5 in [8]).

To measure the fairness and the performance, we employ two criteria that have been used frequently in the past. For instance, it is customary to evaluate the fairness with the following max-min criterion: a rate vector is called max-min fair (MMF) if and only if an increase in the rate of one user results in the decrease in the rate of one or more users who have smaller or equal rates [1, 9]. Additionally, we consider the AME. Note that AME quantifies the loss of performance when the interferer users are present and the background noise vanishes [2–4]. More precisely, AME is a measure of degradation in bit error rate because of the presence of multiple-access interference in a white Gaussian channel.

1.2. Basics of the New Method. Our approach exploits the beneficial aspects of both OMA and SIC. Because we do not aim to improve fairness by sacrificing the throughput, we assume that (2) is satisfied with equality.

The key idea is to partition the network into (almost) homogeneous subnetworks such that the users within each subnetwork employ OMA, and SIC is utilized across subnetworks. Since OMA is applied to (almost) homogeneous subnetworks, it is likely that the degree of fairness is not deteriorated. The application of SIC to subnetworks and not directly to users allows to decrease the number of decoding stages, which potentially improves the performance.

Given that the number of users is K, we assume that the number of subnetworks is $T \in \{1, \ldots, K\}$. The newly proposed scheme is equivalent to partition the K users into T ordered groups. Note that the order matters because it corresponds to the order in which the groups are decoded. Remark for $T = 1$ that the grouping method is the same with OMA. Moreover, the grouping method is identical with SIC for $T = K$. Similarly to conventional SIC, the max-min rate achieved in this case depends on the order in which the groups are decoded. We consider the family of all *ordered partitions* of the K users into T nonempty groups. Then we pick up the ordered partition for which the minimum rate is maximized, and we name it $\text{BORG}_{K/T}$ (*basic ordered grouping of K users into T groups*). Conventionally, $\text{BORG}_{K/1}$ coincides with OMA, and we write $\text{BORG}_{K/1} \equiv \text{OMA}$. Obviously, $\text{BORG}_{K/K} \equiv \text{SIC}$.

Furthermore, one can select again from $\text{BORG}_{K/1}$, $\text{BORG}_{K/2}, \ldots, \text{BORG}_{K/K}$ the ordered partition which maximizes the minimum rate. The new selection is dubbed BORG_K^*. Remark that the rate vector which corresponds to BORG_K^* is not necessarily the same with the max-min fair rate vector that was defined in Section 1.1. However, BORG_K^* is guaranteed to be max-min fair among all possible user groupings for which the sum capacity is achieved.

We investigate how the fairness can be evaluated for OMA, SIC, and BORG. In this context we demonstrate for $\text{BORG}_{K/T}$ a fundamental property, which allows us to introduce a low-complexity search method for choosing $\text{BORG}_{K/T}$ from all ordered partitions of K users into T groups.

We give also a geometrical interpretation for the rate-vector yield by our algorithm. More exactly, we point out the connections between the outcome of the proposed method and the polymatroid structure of the capacity region as it is used in multiuser information theory [1, 10, 11].

During recent years, several works have exploited the polymatroid structure for optimizing the fairness in multi-access systems [1, 12]. The main idea is based on the fact that particular points within the sum capacity facet of the polymatroid can be achieved by successive decoding and time sharing. Then, the effort is focused on finding the time-sharing coefficients which give the fairest point. For the sake of comparison, we pick up the method from [1], which is based on time sharing, and we name it TS. It is clear that TS cannot be inferior to our method if the criterion is the fairness in the multiaccess system. But since TS is a linear combination of successive decoders with different decoding orders, it suffers from the same deficiencies like the ones mentioned earlier for SIC.

The rest of the paper is organized as follows. Section 2 contains the main contribution, where we show how a low-complexity search algorithm can be devised to find the ordered partition which maximizes the minimum rate. The geometrical interpretation of the result is included. In Section 3, the newly proposed method is compared with OMA, SIC, and TS in a simulation study which comprises four different network models. In all cases, the new strategy provides the best tradeoff between fairness and AME.

2. Fairness

2.1. Formulas for OMA and SIC. It is well known for OMA method that the degree of fairness among users is lowered when their received powers are very disparate [5]. This drawback can be easily understood from the formula which gives the rate of the kth user [5, Chapter 6]:

$$R_k^{\text{OMA}} = \frac{P_k}{\Pi}\left[\log\left(1 + \frac{\Pi}{N_0}\right)\right], \qquad (4)$$

where $\Pi = \sum_{j=1}^K P_j$. From the identity above, we have $R_j^{\text{OMA}}/R_k^{\text{OMA}} = P_j/P_k$ for all $j, k \in \mathcal{G}$, where $\mathcal{G} = \{1, \ldots, K\}$. Hence, the rates are as disparate as the received powers are, which leads to unfair rates in heterogeneous networks. For example, if $\min_{1 \le k \le K} P_k/\Pi \to 0$, then the minimum rate $\min_{1 \le k \le K} R_k^{\text{OMA}}$ tends also to zero.

When SIC is applied, the fairest rate vector is obtained by decoding the users in the decreasing order of their received powers [1]. Consequently, the rate of the kth user has the expression [5]

$$R_k^{\text{SIC}} = \log\left(1 + \frac{P_k}{N_0 + \sum_{\{j \mid P_j < P_k\}} P_j}\right). \tag{5}$$

With the convention that $n_k \in \{0, \dots, K - 1\}$ denotes the number of users whose received power is smaller than P_k, the following inequality is readily obtained: $R_k^{\text{SIC}} > \log(1 + 1/(N_0/P_k + n_k))$. It shows that, as long as $\min_{1 \leq k \leq K} P_k$ is not much smaller than N_0, then $\min_{1 \leq k \leq K} R_k^{\text{SIC}}$ does not tend to zero when $\min_{1 \leq k \leq K} P_k/\Pi \to 0$. Hence, it is likely that SIC has a higher degree of fairness than OMA when the received powers are very disparate.

2.2. BORG and Its Low-Complexity Implementation.

Consider the following scenario: K users are divided into nonempty groups $\mathcal{G}_1, \dots, \mathcal{G}_T$. For an arbitrary $i \in \{1, \dots, T\}$, we use Π_i to denote the sum of the received powers for the users that belong to the group $\mathcal{G}_i$. It is clear that $\mathcal{G} = \bigcup_{i=1}^T \mathcal{G}_i$ and $\Pi = \sum_{i=1}^T \Pi_i$.

To be in line with the previous literature, we adopt the convention that the order of the groups in the successive decoding is $\mathcal{G}_{\sigma(T)}, \dots, \mathcal{G}_{\sigma(1)}$, where $\sigma(\cdot)$ is a permutation of the set $\{1, \dots, T\}$. For simplicity, we denote by ORG the ordered partition which is given by the sequence of subsets $\mathcal{G}_{\sigma(T)}, \dots, \mathcal{G}_{\sigma(1)}$. By combining the results from (4) and (5), we get the rate of the kth user

$$R_{k \mid \sigma(t)}^{\text{ORG}} = \frac{P_k}{\Pi_{\sigma(t)}} \left[\log\left(1 + \frac{\Pi_{\sigma(t)}}{N_0 + \sum_{\{j \mid 1 \leq j < t\}} \Pi_{\sigma(j)}}\right)\right]. \tag{6}$$

For writing the equation above more compactly, we have assumed that the kth user belongs to the group $\mathcal{G}_{\sigma(t)}$.

The naive approach for finding $\text{BORG}_{K/T}$ when $T \in \{2, \dots, K - 1\}$ is to search among all ordered partitions of the K users into T groups, then to compute the minimum rate in each case, and eventually to pick up the partition which maximizes the minimum rate. This leads to a huge computational burden and makes the method unpractical. We show below how the number of ordered partitions to be considered can be reduced significantly.

We need some more definitions. Let $\bar{\kappa} = [\kappa_1, \dots, \kappa_T]$ be a vector of strictly positive integers whose sum is equal to K. The ordered partition $\mathcal{G}_{\sigma(T)}, \dots, \mathcal{G}_{\sigma(1)}$ is of type $\bar{\kappa}$ if for all $i \in \{1, \dots, T\}$ the cardinality of $\mathcal{G}_{\sigma(i)}$ is κ_i. Given $\sigma(\cdot)$ and $\bar{\kappa}$, we denote by $\text{ORG}_{K/T}^{\sigma(\cdot), \bar{\kappa}}$ the family of all ordered partitions of type $\bar{\kappa}$. It is important to remark that for all partitions within this family we have that (i) the order of the subsets is the same and is given by the reverse order of the permutation $\sigma(\cdot)$; (ii) the cardinality of the ith subset is the same, namely, κ_i.

Additionally, for two arbitrary subsets $\mathcal{G}_i$ and $\mathcal{G}_j$, we write $\mathcal{G}_i \succ \mathcal{G}_j$ if the received powers of all users within $\mathcal{G}_i$ are greater than those of the users within $\mathcal{G}_j$. When the condition is not satisfied, we write $\mathcal{G}_i \not\succ \mathcal{G}_j$.

Theorem 1. *Let $T \in \{2, \dots, K - 1\}$. For fixed $\sigma(\cdot)$ and $\bar{\kappa}$, consider all ordered partitions that belong to $\text{ORG}_{K/T}^{\sigma(\cdot), \bar{\kappa}}$. In this class, the ordered partition that maximizes the minimum rate for a given set $\{P_1, \dots, P_K\}$ is the one which satisfies the condition*

$$\mathcal{G}_{\sigma(T)} \succ \cdots \succ \mathcal{G}_{\sigma(1)}. \tag{7}$$

The proof is deferred to the appendix.

Now we are prepared to formalize the result which shows the decrease in computational complexity.

Corollary 2. *For $T \in \{2, \dots, K - 1\}$, we have the following.*

(i) *To select $\text{BORG}_{K/T}$ by brute-force search amounts to compute the minimum rate for*

$$\sum_{t=0}^T \binom{T}{t} (-1)^t (T - t)^K \tag{8}$$

different ordered partitions.

(ii) *Theorem 1 allows to reduce to*

$$\binom{K - 1}{T - 1} \tag{9}$$

the number of ordered partitions that are considered in the evaluation process.

Proof. (i) In the case of brute-force search, it is easy to note that the rate vector must be computed for all ordered partitions of the K users into T nonempty subsets. Hence, the number of partitions to be considered equals $T! \times \left\{ {K \atop T} \right\}$, where $\left\{ {K \atop T} \right\}$ is the Stirling number of the second kind, and its closed-form expression is given by $(1/T!) \sum_{t=0}^T \binom{T}{t} (-1)^t (T - t)^K$ [13]. This proves the result in (8).

(ii) From Theorem 1, we know that for all permutations $\sigma(\cdot)$ there exists a single ordered partition of type $\bar{\kappa}$ that must be considered, namely, the one which satisfies (7). For finding $\text{BORG}_{K/T}$, we must evaluate a single rate vector for each vector type. This implies that the number of partitions which are investigated equals the number of ways that the integer K can be written as a sum of T strictly positive integers. According to [13], this number is given by (9). □

To gain more insight, let us suppose that $K = 10$ users and $T = 3$ groups. Corollary 2 points out that the number of competing partitions for the selection of $\text{BORG}_{K/T}$ can be reduced from 55980 to 36, which implies a significant decrease of the computational complexity. However, by using the result from (9), it is easy to verify that the number of rate vectors which must be evaluated for selecting BORG_K^* is 2^{K-1}.

2.3. Geometrical Interpretation.

We resort to the polymatroid structure of the capacity region [1, 10, 11], to give a

new interpretation of BORG. The object of interest is the polyhedron defined by

$$\mathcal{B}(f,\mathcal{G}) = \left\{ (R_1,\ldots,R_K) \in \mathbb{R}_+^K \mid \sum_{j \in \mathcal{S}} R_j \le f(\mathcal{S}), \ \forall \mathcal{S} \subset \mathcal{G} \right\}, \tag{10}$$

where the set function $f : 2^{\mathcal{G}} \to \mathbb{R}_+$ is a mapping for all subsets of $\mathcal{G}$ to the positive real numbers. For the problem that we study, it is convenient to choose

$$f(\mathcal{S}) = \log\left(1 + \frac{\sum_{j \in \mathcal{S}} P_j}{N_0} \right). \tag{11}$$

Then, it is a simple exercise to verify that the set function defined in (11) satisfies (i) $f(\varnothing) = 0$ (normalized); (ii) $f(\mathcal{S}_1) \le f(\mathcal{S}_2)$ if $\mathcal{S}_1 \subset \mathcal{S}_2$ (increasing); (iii) $f(\mathcal{S}_1) + f(\mathcal{S}_2) \ge f(\mathcal{S}_1 \cap \mathcal{S}_2) + f(\mathcal{S}_1 \cup \mathcal{S}_2)$ (submodular). Thus, according to the definition from [14], $\mathcal{B}(f,\mathcal{G})$ is a *polymatroid*. Moreover, the hyperplane given by $\sum_{j=1}^{K} R_j = f(\mathcal{G})$ is the *sum-capacity facet* of $\mathcal{B}(f,\mathcal{G})$ [1]. We analyze next the points within the sum-capacity facet that correspond to OMA, SIC, and ORG. To get the point which corresponds to OMA, we rewrite the identity in (4) as

$$R_k^{\text{OMA}} = \frac{P_k}{\Pi} f(\mathcal{G}). \tag{12}$$

For SIC, we take $\varphi(\cdot)$ to be an arbitrary permutation of the set $\{1,\ldots,K\}$, and we assume that the users are decoded in the reverse order of $\varphi(\cdot)$. Therefore, the rate of the $\varphi(i)$th user is

$$R_{\varphi(i)}^{\text{SIC}} = \begin{cases} f(\{\varphi(i)\}), & i = 1 \\ f(\{\varphi(1),\ldots,\varphi(i)\}) \\ \quad -f(\{\varphi(1),\ldots,\varphi(i-1)\}), & i \in \{2,\ldots,K\}. \end{cases} \tag{13}$$

The formula in (5) is easily obtained from (13) for the particular case when $\varphi(\cdot)$ is chosen such that $P_{\varphi(K)} > \cdots > P_{\varphi(1)}$. More importantly, (13) shows that, for each permutation $\varphi(\cdot)$, $(R_{\varphi(1)},\ldots,R_{\varphi(K)})$ is a corner point of the polymatroid $\mathcal{B}(f,\mathcal{G})$ (see [10, 14] for more details).

With the convention that the kth user belongs to $\mathcal{G}_{\sigma(t)}$, where $\sigma(\cdot)$ is a permutation of $\{1,\ldots,T\}$, the expression in (6) is equivalent to

$$R_{k|\sigma(t)}^{\text{ORG}} = \begin{cases} \dfrac{P_k}{\Pi_{\sigma(t)}} f(\mathcal{G}_{\sigma(t)}), & t = 1 \\ \dfrac{P_k}{\Pi_{\sigma(t)}} f\left(\bigcup_{j=1}^{t} \mathcal{G}_{\sigma(j)} \right) \\ \quad -\dfrac{P_k}{\Pi_{\sigma(t)}} f\left(\bigcup_{j=1}^{t-1} \mathcal{G}_{\sigma(j)} \right), & t \in \{2,\ldots,T\}. \end{cases} \tag{14}$$

Observe that, in general, the rate vector given by (14) does not correspond to a corner of $\mathcal{B}(f,\mathcal{G})$.

To enhance intuition, we depict in Figure 1(a) the polymatroid $\mathcal{B}(f,\mathcal{G})$ for $K = 2$ when the two users have equal received powers ($P_1 = P_2$). Similarly, in Figure 1(b), it is shown $\mathcal{B}(f,\mathcal{G})$ for the case when $P_1 \ll P_2$. In both cases, the sum-capacity facet is the segment whose endpoints are the corners SIC_{1-2} and SIC_{2-1}.

Because $K = 2$, the number of groups for the BORG method can be either $T = 1$ or $T = 2$. Thus, we are interested in the points within the sum-capacity facet that correspond to $\text{BORG}_{2/1}$ and $\text{BORG}_{2/2}$, respectively. As we already know, $\text{BORG}_{2/1}$ is the same with OMA, and $\text{BORG}_{2/2}$ coincides with SIC, where $\text{SIC} \equiv \text{SIC}_{i-j}$ with $P_i > P_j$. Consequently, BORG_2^* is chosen by selecting between OMA and SIC the one which maximizes the minimum rate. For completeness, we consider also the point TS that corresponds to the degree of fairness provided by the method from [1], which finds optimum weights for the time sharing between SIC_{1-2} and SIC_{2-1}.

Note in Figure 1(a) that the OMA point is the fairest on the sum-capacity facet. In this case, it is obvious that also BORG_2^* corresponds to the fairest point. Moreover, the method from [1] gives the same weight to both SIC_{1-2} and SIC_{2-1}, which makes the TS point coincide with the OMA point. The situation changes in Figure 1(b), where $\text{SIC} \equiv \text{SIC}_{2-1}$ and $\min_{1 \le k \le 2} R_k^{\text{SIC}} > \min_{1 \le k \le 2} R_k^{\text{OMA}}$. This leads to $\text{BORG}_2^* \equiv \text{SIC}_{2-1}$. Remark also in Figure 1(b) that, even if BORG_2^* is the best among OMA and SIC, the minimum of its rate vector is slightly smaller than the minimum rate for TS.

Next, we demonstrate by simulations the capabilities of various MA schemes.

3. Simulation Results

3.1. Evaluation Criteria. As it was already mentioned, the fairest rate vector is obtained by applying TS or, equivalently, by time sharing between the corner points of the sum-capacity facet. To find the fairest rate vector and also the optimal time-sharing coefficients, we have implemented in Matlab the algorithms III and IV from [1].

Let us assume that the number of runs for a specified set of experimental conditions is $\mathfrak{N}_r$. An arbitrary method, say MET, is compared with TS by computing the *Normalized Min-Rate* with formula

$$\frac{1}{\mathfrak{N}_r} \sum_{j=1}^{\mathfrak{N}_r} \frac{\min_{1 \le i \le K} R_i^{\text{MET}}(j)}{\min_{1 \le i \le K} R_i^{\text{TS}}(j)}, \tag{15}$$

where $\min_{1 \le i \le K} R_i^{\text{MET}}(j)$ is the minimum of the rate-vector yield by MET in the jth run. Similarly, $\min_{1 \le i \le K} R_i^{\text{TS}}(j)$ is the minimum of the TS rate vector in the jth run.

The second figure of merit that we consider for evaluating the MA schemes is the AME, which is generally denoted by η. AME takes values in the interval $[0,1]$ and attains its maximum when OMA is utilized. Therefore, we have $\eta_k^{\text{OMA}} = 1$ for all $k \in \{1,\ldots,K\}$ (see Chapter 5 in [3]).

To keep SIC inline with what we have in the corner points of the sum-capacity facet, we assume that all cancellations are perfect, and what is forwarded to the next decoder
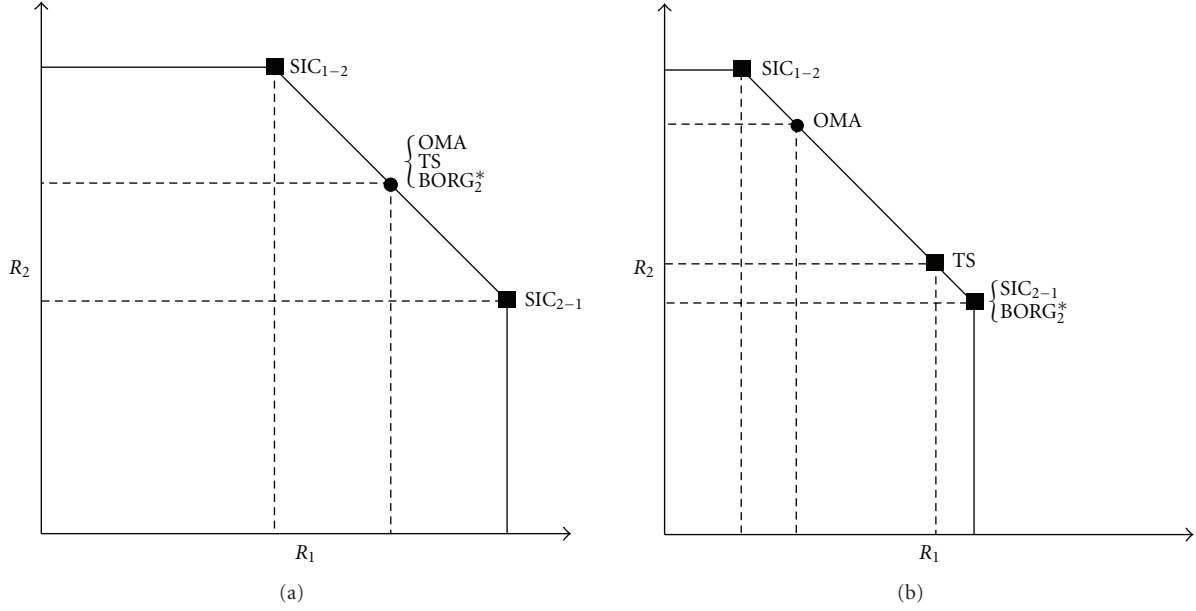
FIGURE 1: The polymatroid $\mathcal{B}(f, \mathcal{G})$ when $K = 2$ users. The points marked on the sum-capacity facet correspond to various MA methods. Here, SIC_{i-j} means that user i is decoded before user j. Two different cases are considered: (a) homogeneous network ($P_1 = P_2$), (b) heterogeneous network ($P_1 \ll P_2$).

has no residual error from the already decoded users [15]. Furthermore, suppose that in each step a matched filter is used as decoder such that for computing the AME of the kth user we can apply the formula (3.123) from [3]:

$$\eta_k^{\text{SIC}} = \left(\max\left\{ 0, 1 - \frac{\sum_{\{j | P_j < P_k\}} P_j^{1/2}}{P_k^{1/2}} \right\} \right)^2. \quad (16)$$

Note that the expression above takes into consideration the system model from (1). Additionally, it is assumed that the users are decoded in the decreasing order of the received powers. We emphasize that we do not use formula (7.31) from [3] because it was derived for SIC with residual errors propagated from previous steps.

It is clear that, for $T \in \{2, \ldots, K - 1\}$, we do not need to compute AME for all ordered partitions of the K users into T subgroups but only for $\text{BORG}_{K/T}$. With slight abuse of notation, we assume that $\text{BORG}_{K/T}$ is the ordered partition $\mathcal{G}_{\sigma(T)}, \ldots, \mathcal{G}_{\sigma(1)}$, where $\mathcal{G}_{\sigma(T)} \succ \cdots \succ \mathcal{G}_{\sigma(1)}$.

More importantly, the BORG method combines the features of both OMA and SIC such that (i) the users within each group are orthogonal one to each other; (ii) the groups share the entire channel. It is evident that only the second characteristic determines the degradation of the AME. Hence, the expression of AME can be derived straightforwardly from (16) by taking into account that, when decoding the group $\mathcal{G}_{\sigma(t)}$, the role of interferer is played by the groups $\mathcal{G}_{\sigma(t-1)}, \ldots, \mathcal{G}_{\sigma(1)}$. If the kth user belongs to the group $\mathcal{G}_{\sigma(t)}$, then we have

$$\eta_{k|\sigma(t)}^{\text{BORG}} = \left(\max\left\{ 0, 1 - \frac{\sum_{\{j | 1 \le j < t\}} \Pi_{\sigma(j)}^{1/2}}{\Pi_{\sigma(t)}^{1/2}} \right\} \right)^2. \quad (17)$$

Remark in the expression above that AME is the same for all the users within the $\mathcal{G}_{\sigma(t)}$-group.

It is worth mentioning here that $\text{BORG}_{K/T}$ does not necessarily coincide with the grouping that maximizes the AME. For example, if the received powers are $P_1 = 100$, $P_2 = 90$, $P_3 = 85$, $P_4 = 15$, $P_5 = 6$, $P_6 = 5$, $P_7 = 4$, $P_8 = 2$, and $P_9 = 1$, then the optimum AME is produced by the ordered partition $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, where $\mathcal{G}_1 = \{1, 2, 3, 5\}$, $\mathcal{G}_2 = \{4, 6, 7\}$, and $\mathcal{G}_3 = \{8, 9\}$. The inequality $P_4 > P_5$ implies $\mathcal{G}_1 \nsucc \mathcal{G}_2$, which shows clearly that $\text{BORG}_{9/3}$ cannot be the ordered partition $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$.

We conclude the short discussion on the second figure of merit, by noticing that, whenever an experiment is repeated $\mathfrak{N}_r$ times, we calculate for each method MET the *Average AME*

$$\frac{1}{\mathfrak{N}_r} \frac{1}{K} \sum_{j=1}^{\mathfrak{N}_r} \sum_{i=1}^{K} \eta_i^{\text{MET}}(j), \quad (18)$$

where $\eta_i^{\text{MET}}(j)$ is the AME for the ith user in the jth run.

In the examples outlined below, the Normalized Min-Rate and the Average AME are employed to compare the performance of the following MA schemes: TS, OMA $\equiv$ $\text{BORG}_{K/1}$, $\text{BORG}_{K/2}$, $\text{BORG}_{K/3}$, SIC $\equiv$ $\text{BORG}_{K/K}$, and BORG_K^*. In our settings, the number of users is $K = 10$, and the number of runs for each set of experimental conditions is $\mathfrak{N}_r = 10^4$. Additionally, the power of the Gaussian noise is taken to be one ($N_0 = 1$). Four different network models are considered.

3.2. Examples

Model I. To quantify the degree of network heterogeneity, we consider the ratio between the power of the

strongest user and the power of the weakest user: $\Delta^{(I)} = \max_{1 \leq i \leq K} P_i / \min_{1 \leq i \leq K} P_i$. The larger is $\Delta^{(I)}$, the more heterogeneous is the network. For a fixed value $\Delta^{(I)} > 1$, we take $P_1 = 200\Delta^{(I)}/(1 + \Delta^{(I)})$ and $P_K = 200/(1 + \Delta^{(I)})$. The powers $P_2, \ldots, P_{K-1}$ are chosen to be outcomes from a uniform distribution on (P_K, P_1), and the experiment is repeated $\mathfrak{N}_r$ times. This selection guarantees that the mean power $\mathbb{E}[P_i]$ is equal to 100. When $\Delta^{(I)} = 1$, a single realization is considered, namely, $P_1 = \cdots = P_K = 100$.

We plot in Figure 2(a) the Normalized Min-Rate obtained for various MA schemes when $\Delta^{(I)}$ increases from 0 dB to 30 dB. Due to the definition in (15), the graph for TS is a straight line parallel to x-axis. Note in the same figure that the degree of fairness is very high for OMA when $\Delta^{(I)}$ is close to 0 dB, but it decreases rapidly when the heterogeneity of the network increases. By contrast, SIC has a very low degree of fairness in homogeneous networks, but it improves with the increase of the network heterogeneity such that for $\Delta^{(I)} > 15$ dB, SIC is clearly superior to OMA.

The beneficial effects of the newly proposed strategy can be observed for BORG_K^*, which performs as well as OMA for small $\Delta^{(I)}$, but surpasses both OMA and SIC for large $\Delta^{(I)}$. More interestingly, good results are obtained not only when searching for the optimum T, but also when the number of groups is kept fixed. Remark for the heterogeneous networks that $\text{BORG}_{K/3}$ performs very similarly with BORG_K^*.

In Figure 2(b), we show the Average AME for the six methods which are compared. As it was already pointed out previously, OMA achieves always the maximum possible AME. We can notice from Figure 2(b) that SIC and TS yield the poorest Average AME. This drawback appears for the two schemes because the firstly decoded users receive high interference, which makes their AME close to zero. It is remarkable that $\text{BORG}_{K/2}$ and $\text{BORG}_{K/3}$ have AME superior to that of SIC. Moreover, the performance of $\text{BORG}_{K/2}$ approaches the Average AME of OMA when $\Delta^{(I)}$ increases.

Model II. Let $\overline{P} = 100$, $\overline{P}_{c1} = \overline{P} + \Delta^{(II)}/2$, $\overline{P}_{c2} = \overline{P} - \Delta^{(II)}/2$, and $\delta = 5$. We simulate a network such that $P_1, \ldots, P_{K/2}$ are uniformly distributed on $(\overline{P}_{c1} - \delta/2, \overline{P}_{c1} + \delta/2)$, and $P_{K/2+1}, \ldots, P_K$ are uniformly distributed on $(\overline{P}_{c2} - \delta/2, \overline{P}_{c2} + \delta/2)$. The parameter $\Delta^{(II)}$ controls the degree of heterogeneity of the network. When $\Delta^{(II)} = 0$, all the users belong to a single cluster, and the increase of $\Delta^{(II)}$ makes the network to consist of two disjoint clusters.

The results plotted in Figure 3 are obtained by varying the value of $\Delta^{(II)}$ from 0 to 180. For each value of $\Delta^{(II)}$, $\mathfrak{N}_r$ different realizations of $P_1, \ldots, P_K$ are generated. In terms of fairness and AME, the performance of BORG_K^* is the same with that of OMA $\equiv \text{BORG}_{K/1}$ for $\Delta^{(II)} < 100$. Remark in Figure 3(a) that the graph of BORG_K^* coincides with that of $\text{BORG}_{K/2}$ when $\Delta^{(II)}$ is larger than 100. A similar fact can be also observed in Figure 3(b). So, BORG_K^* automatically adapts to the topology of the network.

Model III. We consider again a network including two clusters. This time, the received powers of the users are generated as suggested in [16]. Let $\beta_1 > \beta_2$ such that $\beta_1 + \beta_2 =$
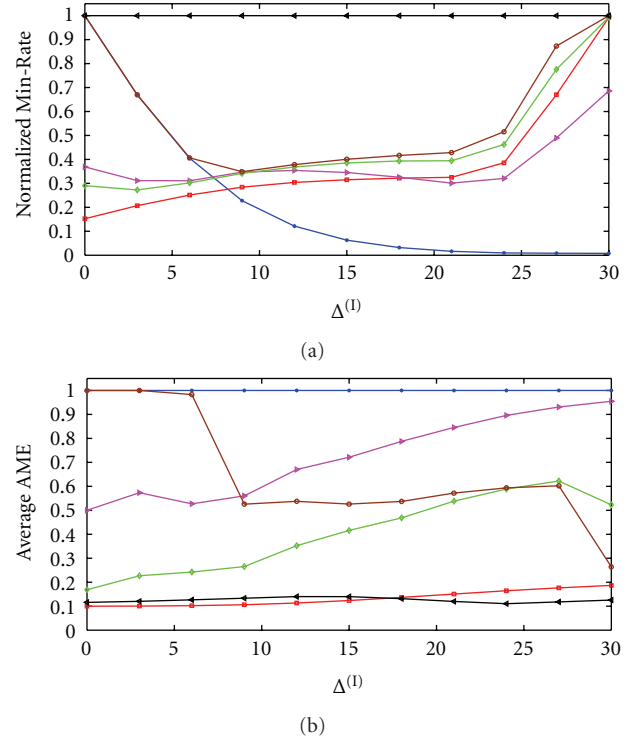

(a)

(b)

FIGURE 2: Experimental results for Model I: (a) Normalized Min-Rate versus $\Delta^{(I)}$; (b) Average AME versus $\Delta^{(I)}$. The number of users is $K = 10$. Note that $\Delta^{(I)}$ is expressed in dB, and for each $\Delta^{(I)} > 0$ the reported results are obtained from $\mathfrak{N}_r = 10^4$ runs. The following MA methods are compared (for each method we indicate the color and the marker symbol used in plots): TS (black left-pointing triangle), OMA $\equiv \text{BORG}_{K/1}$ (blue asterisk), $\text{BORG}_{K/2}$ (magenta right-pointing triangle), $\text{BORG}_{K/3}$ (green diamond), SIC $\equiv \text{BORG}_{K/K}$ (red square), and BORG_K^* (brown circle).

100. We take $P_i = \beta_j z_i$, where $j = 1$ if $i \in \{1, \ldots, K/2\}$ and $j = 2$ if $i \in \{K/2 + 1, \ldots, K\}$. The distribution of the random variable z_i is Chi-Square with two degrees of freedom.

Remark that the heterogeneity of the network is measured by the difference $\Delta^{(III)} = \beta_1 - \beta_2$, which we increase from 0 to 90. The Normalized Min-Rate and the Average AME calculated for each $\Delta^{(III)}$ based on $\mathfrak{N}_r$ runs are shown in Figure 4. It is easy to observe the following outcome of the experiment. Because the Chi-Square distribution has infinite support, OMA does not provide fairness in rate allocation when $\Delta^{(III)} = 0$. Due to the same reason, for all values of $\Delta^{(III)}$, the degree of fairness yield by $\text{BORG}_{K/2}$ is inferior to that of $\text{BORG}_{K/3}$ even if, for $\text{BORG}_{K/2}$, the number of groups equals the "true" number of clusters. However, when comparing the Average AME, $\text{BORG}_{K/2}$ is ranked the second after OMA which achieves the maximum possible value.

Model IV. Following the suggestion of one of the reviewers, we briefly investigate the case of K users uniformly distributed over a two-dimensional area. For the sake of concreteness, let $d_1, \ldots, d_K$ be the distances from the BS to the users. According to the large-scale model, we have $P_i = P_0 d_i^{-n}$, where P_i is the received power from the ith user, P_0 is
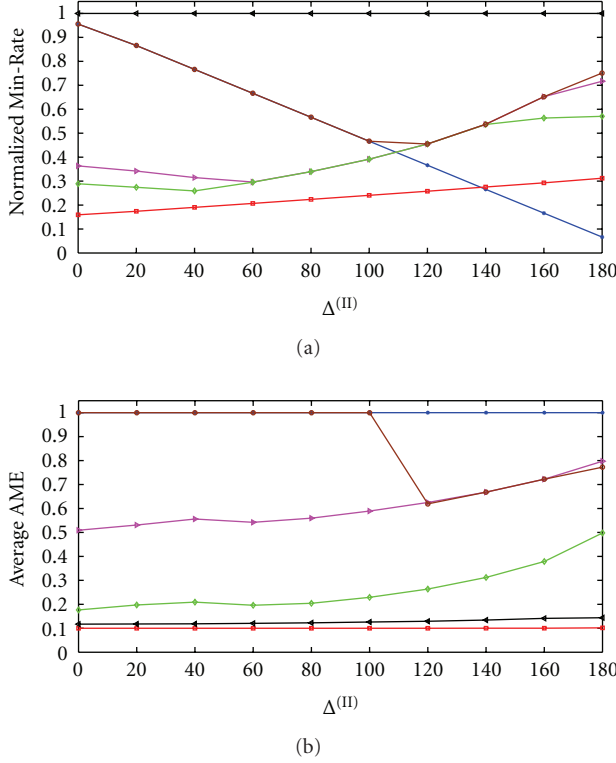
(a)

(b)

FIGURE 3: Experimental results for Model II: (a) Normalized Min-Rate versus $\Delta^{(\mathrm{II})}$; (b) Average AME versus $\Delta^{(\mathrm{II})}$. Note that $\Delta^{(\mathrm{II})}$ is not expressed in dB. The number of users (K), the number of runs ($\mathfrak{N}_r$), and all graphical conventions are the same like in Figure 2.

(a)

(b)

FIGURE 4: Experimental results for Model III: (a) Normalized Min-Rate versus $\Delta^{(\mathrm{III})}$; (b) Average AME versus $\Delta^{(\mathrm{III})}$. Note that $\Delta^{(\mathrm{III})}$ is not expressed in dB. The number of users (K), the number of runs ($\mathfrak{N}_r$), and all graphical conventions are the same like in Figure 2.

the received power from a transmitter located at distance one from the BS, d_i is the distance from the ith user to the BS, and n is the *path loss exponent* [17, 18].

It is widely accepted that $2.7 < n < 3.5$ for urban area cellular radio [18, Table 3.2]. In our settings, we choose the path loss exponent to be $n = 3$ and $P_0 = 2700$. For two arbitrary bounds δ_ℓ and δ_u with property $0 < \delta_\ell < \delta_u$, the squared distances $d_1^2, \ldots, d_K^2$ are selected to be uniformly distributed on $(\delta_\ell^2, \delta_u^2)$. Hence, the mean power has the expression $\mathbb{E}[P_i] = 2P_0/(\delta_\ell^2 \delta_u + \delta_\ell \delta_u^2)$. Let us consider various values of δ_ℓ between 1.2 and 3.0, and for each δ_ℓ we choose δ_u such that $\mathbb{E}[P_i] = 100$. Conventionally we take $\delta_0 = 3.0$. It is easy to verify that $\delta_\ell = \delta_0$ implies $\delta_u = \delta_0$, or equivalently all the users are located on a circle whose center coincides with the BS. Obviously, this corresponds to the case of a homogeneous network. In fact, for all $\delta_\ell \in [1.2, 3.0]$, the quantity $\Delta^{(\mathrm{IV})} = \delta_0 - \delta_\ell$ can be used to measure the heterogeneity of the network: the bigger is $\Delta^{(\mathrm{IV})}$, the larger is the difference $\delta_u^2 - \delta_\ell^2$, which makes the values of P_i, $i \in \{1, \ldots, K\}$, more disparate.

In Figure 5, we plot the Normalized Min-Rate and the Average AME. They are computed for each $\Delta^{(\mathrm{IV})} \in [0.2, 1.8]$ based on $\mathfrak{N}_r$ runs, while for $\Delta^{(\mathrm{IV})} = 0.0$ one single realization is considered. By comparing the results within Figure 5 with those from Figure 2, we can observe that the multiaccess schemes have a similar behavior for Model IV and Model I.
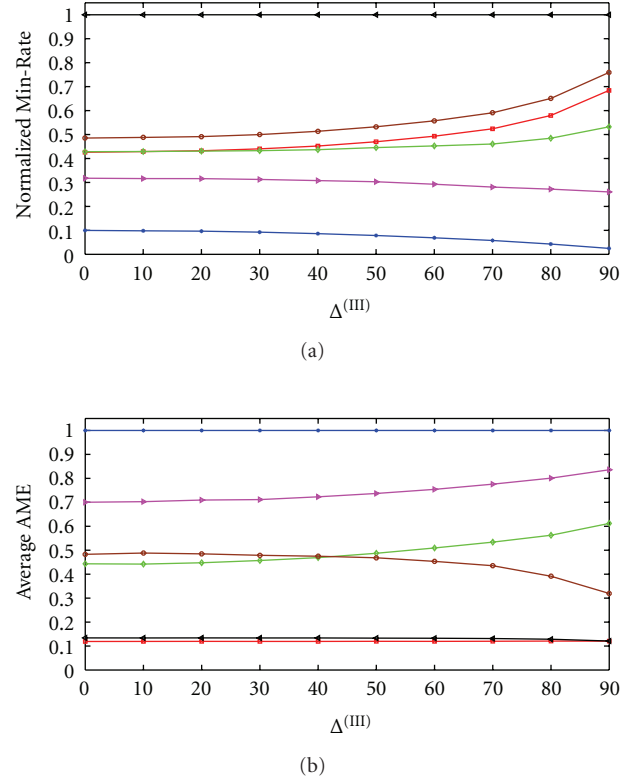
As a final remark, we note that for all four network models, finding the partition which corresponds to BORG_K^* is faster than applying the TS optimization strategy. In all runs, the execution time for BORG_K^* was at about 40% of the execution time for TS. When the number of users is very large, the computational complexity can be decreased by searching for $\mathrm{BORK}_{K/T}$ with a fixed T, instead of finding the partition BORK_K^*. We observe from the numerical examples that $\mathrm{BORK}_{K/2}$ and $\mathrm{BORK}_{K/3}$ have an acceptable level of performance.

4. Conclusion

In this paper, we investigated how OMA and SIC can be combined to improve fairness in Gaussian wireless networks. The newly proposed method divides the network into (almost) homogeneous subnetworks such that the users within each subnetwork employ OMA, and SIC is utilized across subnetworks. Equivalently, the K users are partitioned into T ordered groups. The main theoretical result which we proved for any $T \in \{2, \ldots, K-1\}$ shows that the ordered partition which maximizes the minimum rate can be found with a low-complexity algorithm. Moreover, it was demonstrated experimentally that the user grouping strategy guarantees a good tradeoff between fairness and the asymptotic multiuser efficiency.
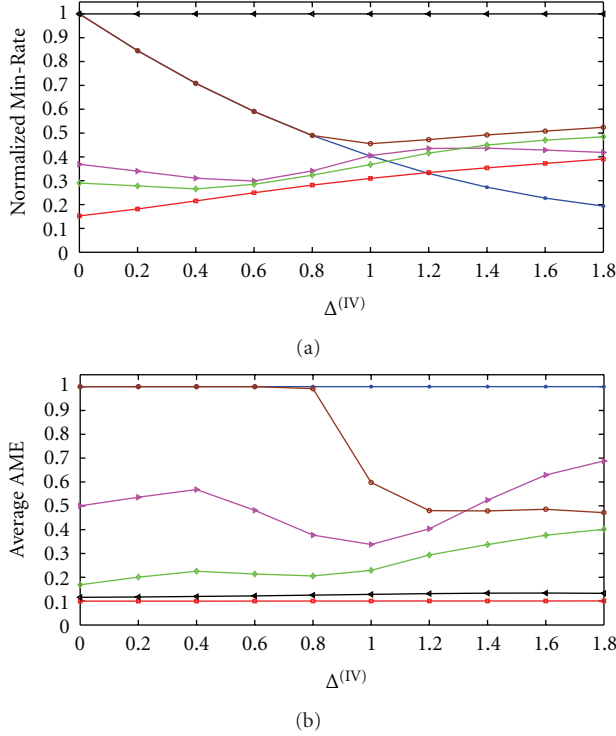
(a)

(b)

FIGURE 5: Experimental results for Model IV: (a) Normalized Min-Rate versus $\Delta^{(IV)}$; (b) Average AME versus $\Delta^{(IV)}$. Note that $\Delta^{(IV)}$ is not expressed in dB. The number of users (K), the number of runs ($\mathfrak{N}_r$), and all graphical conventions are the same like in Figure 2.

Appendix

Proof of Theorem 1

First we demonstrate two auxiliary results that are instrumental in proving Theorem 1.

Lemma 3. *(i) Let $a, b, c > 0$ with $b > a$. The following inequality holds:*

$$\frac{1}{b} \log\left(1 + \frac{b}{c}\right) < \frac{1}{a} \log\left(1 + \frac{a}{c}\right). \tag{A.1}$$

(ii) For $a, b, c, d > 0$ that satisfy simultaneously the conditions $a > b$ and $a + c = b + d$, we have

$$\frac{1}{b} \log\left(1 + \frac{b}{d}\right) < \frac{1}{a} \log\left(1 + \frac{a}{c}\right). \tag{A.2}$$

Proof. For an arbitrary $x > 0$, the following inequality is well known [19]:

$$\frac{x}{1 + x} < \ln(1 + x) < x, \tag{A.3}$$

where $\ln(\cdot)$ denotes the natural logarithm. By using this result together with some elementary calculations, the inequalities in (A.1) and (A.2) are readily obtained.

(i) Since $b > a$, there exists $x > 0$ such that $b = a + x$. So, the left-hand side of (A.1) can be expressed as $h(x) =$

$(1/\ln 2/(a + x))\ln(1 + (a + x)/c)$. It is easy to check that the first derivative of $h(x)$ is strictly negative for all $x > 0$:

$$\frac{dh(x)}{dx} = \frac{1/\ln 2}{a + x}\left[\frac{1}{a + x + c} - \frac{1}{a + x}\ln\left(1 + \frac{a + x}{c}\right)\right]$$

$$< \frac{1/\ln 2}{a + x}\left[\frac{1}{a + x + c} - \frac{1}{a + x}\frac{(a + x)/c}{1 + (a + x)/c}\right] = 0. \tag{A.4}$$

Note that the inequality in (A.4) is a straightforward consequence of (A.3). The fact that $h(x)$ is a monotonically decreasing function proves the inequality in (A.1).

(ii) The inequality $a > b$ guarantees that there exists $x > 0$ such that $b = a - x$. Moreover, we have $d = c + x$ because $a + c = b + d$. For all $x \geq 0$, we define $h(x) = (1/\ln 2/(a - x))\ln(1 + (a - x)/(c + x))$. In (A.2), the left-hand side is equal to $h(x)$ with $x > 0$, whereas the right-hand side coincides with $h(0)$. To prove the inequality, it is enough to show that $h(x)$ is a monotonically decreasing function or, equivalently, to verify that the first derivative of $h(x)$ is strictly negative:

$$\frac{dh(x)}{dx} = \frac{1/\ln 2}{(a - x)^2}\ln\left(1 + \frac{a - x}{c + x}\right) - \frac{1/\ln 2}{(a - x)(c + x)}$$

$$< \frac{1/\ln 2}{(a - x)^2}\frac{a - x}{c + x} - \frac{1/\ln 2}{(a - x)(c + x)} = 0. \tag{A.5}$$

The inequality in (A.5) was obtained by applying (A.3). □

Lemma 4. *For a given set $\{P_1, \ldots, P_K\}$, let $\mathcal{G}_{\sigma(T)}, \ldots, \mathcal{G}_{\sigma(1)}$ and $\mathcal{G}'_{\sigma(T)}, \ldots, \mathcal{G}'_{\sigma(1)}$ be two ordered partitions from $ORG_{K/T}^{\sigma(\cdot), \overline{\kappa}}$. If there exists $i \in \{1, \ldots, T - 1\}$ such that*

(C_1) $\mathcal{G}_{\sigma(j)} = \mathcal{G}'_{\sigma(j)}$ for all $j \in \{1, \ldots, T\} \setminus \{i, i + 1\}$,

(C_2) $\mathcal{G}_{\sigma(i+1)} \not\succ \mathcal{G}_{\sigma(i)}$,

(C_3) $\mathcal{G}'_{\sigma(i+1)} \succ \mathcal{G}'_{\sigma(i)}$,

then the minimum rate corresponding to $\mathcal{G}_{\sigma(T)}, \ldots, \mathcal{G}_{\sigma(1)}$ is not larger than the minimum rate corresponding to $\mathcal{G}'_{\sigma(T)}, \ldots, \mathcal{G}'_{\sigma(1)}$.

Proof. Without loss of generality, we assume $\sigma(j) = j$ for all $j \in \{1, \ldots, T\}$. Additionally, we make the assumption that do not exist two different users for which the received powers are the same. With the exception of the symbol $'$, all the notations employed in connection with the ordered partition $\mathcal{G}'_T, \ldots, \mathcal{G}'_1$ are the same with those utilized for $\mathcal{G}_T, \ldots, \mathcal{G}_1$.

The formula in (6) leads to the following expression of the minimum rate for the users within $\mathcal{G}_j$:

$$R_{\min,j} = \frac{P_{\min,j}}{\Pi_j}\left[\log\left(1 + \frac{\Pi_j}{N_j}\right)\right], \tag{A.6}$$

where $P_{\min,j}$ is the received power of the weakest user within $\mathcal{G}_j$ and $N_j = N_0 + \sum_{t=1}^{j-1} \Pi_t$. The equation above together

with condition (C_1) lead to the identity $R_{\min,j} = R'_{\min,j}$ for all $j \in \{1,\ldots,T\} \setminus \{i,i+1\}$. Hence, for proving Lemma 4, it is enough to show that

$$\min(R_{\min,i}, R_{\min,i+1}) \leq \min\left(R'_{\min,i}, R'_{\min,i+1}\right). \quad (A.7)$$

This inequality can be obtained from the *three results* which are outlined below.

Result 1. $R_{\min,i+1} < R'_{\min,i+1}$.
To verify the inequality, we note that

$$R_{\min,i+1} = \frac{P_{\min,i+1}}{\Pi_{i+1}} \log\left(1 + \frac{\Pi_{i+1}}{N_{i+1}}\right) \quad (A.8)$$

$$< \frac{P'_{\min,i+1}}{\Pi_{i+1}} \log\left(1 + \frac{\Pi_{i+1}}{N_{i+1}}\right) \quad (A.9)$$

$$< \frac{P'_{\min,i+1}}{\Pi'_{i+1}} \log\left(1 + \frac{\Pi'_{i+1}}{N'_{i+1}}\right) \quad (A.10)$$

$$= R'_{\min,i+1}. \quad (A.11)$$

In (A.9), we use the fact that $P_{\min,i+1} < P'_{\min,i+1}$, which is a consequence of (C_1)–(C_3). The inequality in (A.10) is derived by applying (A.2) with $a = \Pi'_{i+1}$, $b = \Pi_{i+1}$, $c = N'_{i+1}$, and $d = N_{i+1}$.

Result 2. If $P_{\min,i+1} < P_{\min,i}$, then $R_{\min,i+1} < R'_{\min,i}$.
First we consider the case $\Pi_{i+1} > \Pi'_i$ for which we get:

$$R_{\min,i+1} = \frac{P_{\min,i+1}}{\Pi_{i+1}} \log\left(1 + \frac{\Pi_{i+1}}{N_{i+1}}\right) \quad (A.12)$$

$$= \frac{P'_{\min,i}}{\Pi_{i+1}} \log\left(1 + \frac{\Pi_{i+1}}{N_{i+1}}\right) \quad (A.13)$$

$$< \frac{P'_{\min,i}}{\Pi'_i} \log\left(1 + \frac{\Pi'_i}{N_{i+1}}\right) \quad (A.14)$$

$$< \frac{P'_{\min,i}}{\Pi'_i} \log\left(1 + \frac{\Pi'_i}{N'_i}\right) \quad (A.15)$$

$$= R'_{\min,i}. \quad (A.16)$$

Because $\mathcal{G}'_i$ contains the weakest users of $\mathcal{G}_i \cup \mathcal{G}_{i+1}$, condition $P_{\min,i+1} < P_{\min,i}$ implies $P_{\min,i+1} = P'_{\min,i}$, which leads to the identity in (A.13). The inequality in (A.14) can be verified by operating in (A.1) the following substitutions: $a = \Pi'_i$, $b = \Pi_{i+1}$, and $c = N_{i+1}$. It is easy to check that $N_{i+1} - N'_i = \Pi_i$. Hence, $N_{i+1} > N'_i$, which proves the inequality in (A.15).
Under the hypothesis $\Pi_{i+1} < \Pi'_i$, we have

$$R_{\min,i+1} = \frac{P'_{\min,i}}{\Pi_{i+1}} \log\left(1 + \frac{\Pi_{i+1}}{N_{i+1}}\right) \quad (A.17)$$

$$< \frac{P'_{\min,i}}{\Pi'_i} \log\left(1 + \frac{\Pi'_i}{N_{i+1} + \Pi_{i+1} - \Pi'_i}\right) \quad (A.18)$$

$$< \frac{P'_{\min,i}}{\Pi'_i} \log\left(1 + \frac{\Pi'_i}{N'_i}\right) \quad (A.19)$$

$$= R'_{\min,i}. \quad (A.20)$$

The identity in (A.17) is the same with the one from (A.13). Then we write the inequality in (A.2) for the particular case when $a = \Pi'_i$, $b = \Pi_{i+1}$, $c = N_{i+1} + \Pi_{i+1} - \Pi'_i$, $d = N_{i+1}$, and we get (A.18). Observe that $c > \Pi_i - \Pi'_i > 0$. Additionally, $\Pi_{i+1} + \Pi_i > \Pi'_i$ implies $N_{i+1} + \Pi_{i+1} - \Pi'_i > N'_i$, which yields the inequality in (A.19).

Result 3. If $P_{\min,i+1} > P_{\min,i}$, then $R_{\min,i} < R'_{\min,i}$. Note that

$$R_{\min,i} = \frac{P_{\min,i}}{\Pi_i} \log\left(1 + \frac{\Pi_i}{N_i}\right) \quad (A.21)$$

$$= \frac{P'_{\min,i}}{\Pi_i} \log\left(1 + \frac{\Pi_i}{N_i}\right) \quad (A.22)$$

$$< \frac{P'_{\min,i}}{\Pi'_i} \log\left(1 + \frac{\Pi'_i}{N_i}\right) \quad (A.23)$$

$$= \frac{P'_{\min,i}}{\Pi'_i} \log\left(1 + \frac{\Pi'_i}{N'_i}\right) \quad (A.24)$$

$$= R'_{\min,i}. \quad (A.25)$$

The identity in (A.22) is obtained by applying the same type of reasoning like the one used to demonstrate (A.13). Then we focus on (A.23), which is proved by choosing $a = \Pi'_i$, $b = \Pi_i$, $c = N_i$ in (A.1). Furthermore, we get (A.24) from $N_i = N'_i = N_0 + \sum_{j=1}^{i-1} \Pi_j$.
The inequalities in (A.11), (A.16), (A.20), and (A.25) lead to (A.7), which concludes the proof of Lemma 4. □

Now we give the proof of the theorem.

Proof of Theorem 1. Given $\{P_1,\ldots,P_K\}$, we consider the ordered partition $\mathcal{G}_{\sigma(T)},\ldots,\mathcal{G}_{\sigma(1)}$ that belongs to $\mathrm{ORG}_{K/T}^{\sigma(\cdot),\bar{\kappa}}$. For an arbitrary $i \in \{1,\ldots,T-1\}$, we define the following elementary transformation, which is denoted by $\mathrm{ET}(i)$:

(i) if $\mathcal{G}_{\sigma(i+1)} \nsucc \mathcal{G}_{\sigma(i)}$, then $\mathcal{G}_{\sigma(T)},\ldots,\mathcal{G}_{\sigma(1)}$ is transformed to another ordered partition from $\mathrm{ORG}_{K/T}^{\sigma(\cdot),\bar{\kappa}}$, say $\mathcal{G}'_{\sigma(T)},\ldots,\mathcal{G}'_{\sigma(1)}$, such that the conditions (C_1) and (C_3) from Lemma 4 are satisfied;

(ii) otherwise, the ordered partition $\mathcal{G}_{\sigma(T)},\ldots,\mathcal{G}_{\sigma(1)}$ remains unchanged.

Assume that we apply $\mathrm{ET}(1)$ to $\mathcal{G}_{\sigma(T)},\ldots,\mathcal{G}_{\sigma(1)}$. The newly obtained ordered partition is then transformed by $\mathrm{ET}(2)$, and the process continues until $\mathrm{ET}(T-1)$ is applied. Due to the definition of $\mathrm{ET}(\cdot)$, the resulting ordered partition is guaranteed to contain the strongest user in the group which is decoded first. If the type-vector $\bar{\kappa}$ has the property that $\kappa_T > 1$ and the second strongest user was not yet included in the group which is decoded first, then we move it to this group by iterating again from $\mathrm{ET}(1)$ to $\mathrm{ET}(T-1)$. When $\kappa_T = 1$, the second strongest user can be moved to the second decoded group by using similar transformations.
Based on the observations above, we remark that each ordered partition from $\mathrm{ORG}_{K/T}^{\sigma(\cdot),\bar{\kappa}}$ can be transformed, in a finite number of steps, to the unique partition from $\mathrm{ORG}_{K/T}^{\sigma(\cdot),\bar{\kappa}}$ which satisfies the condition in (7). Each step consists in

transforming the current ordered partition to a new one by applying ET(i), where $i \in \{1, \dots, T - 1\}$. We conclude the proof by mentioning that, at each step, the minimum rate increases or remains constant (see **Lemma 4**).   □

Acknowledgment

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