Research Article

Characterizing the Path Coverage of Random Wireless Sensor Networks

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Wireless sensor networks are widely used in security monitoring applications to sense and report specific activities in a field. In path coverage, for example, the network is in charge of monitoring a path and discovering any intruder trying to cross it. In this paper, we investigate the path coverage properties of a randomly deployed wireless sensor network when the number of sensors and also the length of the path are finite. As a consequence, Boolean model, which has been widely used previously, is not applicable. Using results from geometric probability, we determine the probability of full path coverage, distribution of the number of uncovered gaps over the path, and the probability of having no uncovered gaps larger than a specific size. We also find the cumulative distribution function (cdf) of the covered part of the path. Based on our results on the probability of full path coverage, we derive a tight upper bound for the number of nodes guaranteeing the full path coverage with a desired reliability. Through computer simulations, it is verified that for networks with nonasymptotic size, our analysis is accurate where the Boolean model can be inaccurate.

1. Introduction

Wireless sensor networks (WSNs) have many applications in security monitoring. In such applications, since it is essential to keep track of all activities within the field, network coverage is of great importance and must be considered in the network design stage.

Path coverage is one of the monitoring examples, where WSNs are deployed to sense a specific path and report possible efforts made by intruders to cross it. In a manual network deployment, the desired level of the path coverage can be achieved by proper placement of the sensors over the area. When it is not possible to deploy the network manually, random deployment, for example, dropping sensors from an aircraft, is used. Due to the randomness of the sensors location, network coverage expresses a stochastic behavior and the desired (full) path coverage is not guaranteed. Thus, a detailed analysis of the random network coverage can be ultimately useful in the network design stage to determine the node density for achieving the desired area/path coverage.

Path coverage by a random network (or barrier coverage which is a relaxed version of the path coverage) has been the focus of some previous work [1–6]. In [1], assuming that a random network is deployed over an infinite area with nodes following a Poisson distribution, authors investigate the path coverage of the network. They first study the path coverage over an infinite straight line when the sensor has a random sensing range. Then, they show that in the asymptotic situation, where the sensing range of the sensors tends to 0 and the node density approaches infinity, the results are extendible to finite linear and curvilinear paths. Further, a path coverage analysis is proposed for a high-density Poisson-distributed network in [2] where sensors have a fixed sensing range. The path coverage analysis of [1, 2] is based on the Boolean model of [7], where a Poisson point process is justified.

Kumar et al. study k-barrier coverage provided by a random WSN in [3]. To this end, they develop a theoretical model revealing the behavior of the network coverage over a long narrow belt. It is assumed that the sensors are spread over the belt according to a Poisson distribution. The authors

propose an algorithm determining whether an area is k-barrier covered or not. Also, they introduce the concepts of weak and strong barrier coverage over the belt and derive the condition on the sensors density guaranteeing the weak barrier coverage.

The focus of [4] is on the strong barrier coverage. First, authors present a condition insuring the strong barrier coverage over a strip where the sensors locations follow a Poisson point process. Then, by considering asymptotic situation (on the network size and number of nodes) and using Percolation theory [8], they determine, with a probability approaching 1, whether the network has a strong barrier coverage or not. Then, they use their analysis to devise a distributed algorithm to build strong barrier coverage over the strip.

In this work, unlike most existing studies which focus on asymptotic setups, we study the path coverage of a finite random network (in terms of both network size and the number of nodes). As a result, the Boolean model is not accurate. Alternatively, the methodology of this work is based on some results from geometric probability. Our focus is on the path coverage for a circle, but extension to other path shapes is briefly discussed.

In the ideal case, all sensors are located exactly on the path. This, however, is not a practical assumption for randomly deployed networks. To consider the inaccuracy of the sensors locations, we assume that sensors are inside a ring containing the circular path. As a result, the portion of the path covered by any given sensor is not deterministic. Moreover, other factors may affect the sensing range of a sensor. Thus, our analysis is not based on a fixed sensing range. Indeed, we first develop a random model for the covered segment of the path by each sensor. Then, we study the distribution of the number of uncovered gaps on the path. The full path coverage is a special case where the number of gaps is zero. This is used to determine a tight bound on the number of active sensors assuring the full path coverage with a desired reliability. Also, we find the probability of having all possible gaps smaller than a given size. This probability reflects the reliability of detecting an intruding object with a known size.

In addition to studying the number of gaps, we present a simplified analysis for deriving the cumulative distribution function (cdf) of the covered part of the path. This simplified analysis is based on using the expected value of the covered part of the path by a sensor instead of considering the precise random model. We observe that the simplified analysis can provide a fairly accurate approximation of the path coverage.

Since our analysis studies the effect of the number of nodes on the path coverage of a finite size network, it can readily be used in the design of practical networks. In fact, using our results, one can determine the number of nodes in the network to satisfy a desired level of coverage. An example is provided.

The paper is organized as follows. Section 2 introduces the network model and defines the problem. Our coverage analysis is presented in Section 3. Section 4 includes computer simulations verifying our analysis. Finally, Section 5 concludes the paper.
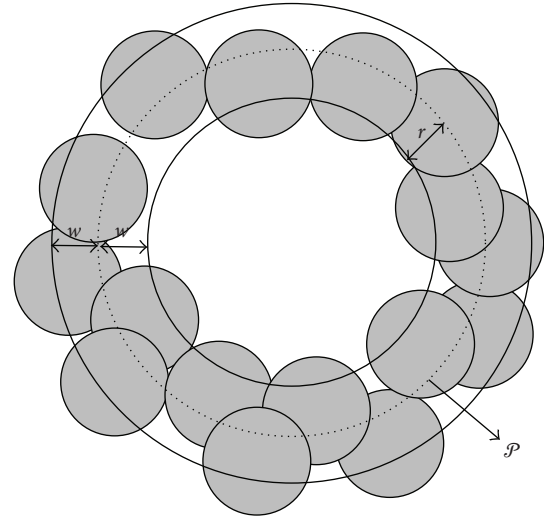

Figure 1: Network structure.

2. Preliminary

2.1. Network Model. We consider N sensors monitoring a circular path with unit circumference, called $\mathcal{P}$. In an ideal case, the sensors are precisely located on the circular path, but this is not usually true for a randomly deployed network. In order to take this fact into account, we assume that sensors are randomly spread over a ring containing $\mathcal{P}$ (See Figure 1). We assume a symmetric distribution for sensors, that is, the sensor density does not depend on the polar angle and is determined only by the distance from the center. It is generally desired to have more sensors in the vicinity of $\mathcal{P}$. Thus, distributions with larger values close to $\mathcal{P}$ are preferred. When no effort is made to put the sensor as close as possible to the path (N sensors are spread totally randomly), the uniform distribution is obtained. Hence, in the sense of placement efforts, uniform distribution reflects the worst case. We consider uniform distribution to verify our analysis by computer simulation in Section 4. Our analysis, however, is presented for any given symmetric distribution. Also, notice that since the number of sensors is finite and known, Poisson distribution, which has been the focus of existing asymptotic analysis, is not applicable.

We also assume that sensors sensing range may vary from r_1 to r_2. Obviously, for a fixed sensing range, model $r_1 = r_2$. Without loss of generality, it is assumed that the width of the ring is smaller than or equal to $2r_2$ and the desired circular path is located at the middle of the ring. Since the sensors farther than r_2 to the path do not contribute to the path coverage, our assumption on the ring width does not hurt the generality of the analysis.

2.2. Motivation and Problem Definition. Our goal in this work is to investigate the quality of the path coverage by the network. To this end, we study the following features of the path coverage which provide us with a concrete insight to the performance of the network.

(i) *Distribution of the number of gaps.* Due to the randomness of the network implementation, sensors may not cover the whole path. In this case, one or more gaps appear. Assume that g represents the number of gaps on $\mathcal{P}$. We are interested to find the probability of having m gaps, shown by $P(g = m)$.

(ii) *Full path coverage.* It is desired to provide a complete coverage of the path. Since the full path coverage is identical to having no gaps, one can equivalently find $P(g = 0)$. This can simply be found from the derived distribution of g.

(iii) *Reliability of the network in detecting objects.* It is important to investigate whether the network is able to detect an object, while the path is not fully covered and there may exist some gaps. Basically, we need to consider the size of the gaps in addition to their number. If one knows the size of the intruders beforehand, it is not necessary to provide the full path coverage. Instead, it is possible to deploy a network such that while the path is not fully covered, the size of the possible gaps is smaller than the intruders. Clearly, implementing a network with possible small gaps requires fewer number of nodes and consequently is less expensive. To this end, we find the probability of having all gaps smaller than a given length t, denoted by $P(l_g < t)$.

(iv) *Distribution of the covered part of the path.* The covered part of the path, $\mathcal{C}$, has a stochastic nature and its distribution provides a general view of the entire path coverage. In fact, the covered part of the network reflects the combined effect of the number of gaps and their sizes. We derive the cdf of $\mathcal{C}$, $F_{\mathcal{C}}(x)$.

3. Path Coverage Analysis

In this section, we present our analysis of the path coverage. For this purpose, we take advantage of existing results in geometric probability and extend them to our case. After the exact coverage analysis, a less complex approximate analysis is also presented.

An arbitrary point on $\mathcal{P}$ is covered if it is within the sensing range of at least one sensor. Here, we assume that the sensing area of sensor i is a circle denoted by s_i, $i = 1, 2, \ldots, N$. The covered part of the path by each s_i is its intersection with $\mathcal{P}$ which is an arc, called a_i. Thus, the total covered part of the path is

$$\mathcal{C} = \bigcup_{i=1}^{N} a_i. \tag{1}$$

Notice that the length of a_i's depends on the location of the sensor within the ring-shaped network area and its sensing range. Considering an arbitrary point as the origin on $\mathcal{P}$ and choosing the clockwise direction as the positive direction, each a_i starts from l_i and continues (clockwise) until r_i, (Figure 2). In other words, l_i determines the most left point of the arc and r_i specifies the most right point of the arc. There are two noteworthy issues here. First, the size
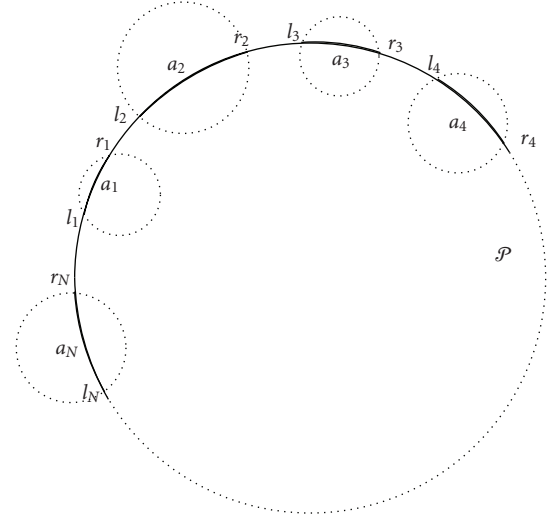

FIGURE 2: The random arcs placed clockwise on $\mathcal{P}$.

of a_i's and their positions are random because of the random placement of the sensors over the ring. Second, $\mathcal{C}$ is not necessarily connected and there may exist several uncovered gaps on $\mathcal{P}$. The number of uncovered gaps on $\mathcal{P}$ and their size can reflect the possible opportunities for intruders to pass $\mathcal{P}$ without being detected by the sensors. If $g = 0$, $\mathcal{P}$ is fully covered.

The problem of covering a circle with random arcs has been studied in geometric probability [9–15]. In some cases, it is assumed that the arcs have a fixed length [9, 12, 13, 15] or the analysis is conducted in the asymptotic situation [10, 14]. Asymptotic analysis is suitable for the situation where the sizes of s_i's are significantly smaller than $\mathcal{P}$. In the following, we initially use the result of [11] on the coverage of a circle with random arcs of random sizes. This helps us to provide an exact explanation of the path coverage. Then, we use the mean value of a_i's to provide a simplified approximate analysis based on the fixed-length random arc placement over the circle [12].

3.1. Exact Analysis. We apply the following theorem from [11] to find the exact distribution of the number of gaps on $\mathcal{P}$.

Theorem 1. *Assume that N arcs are distributed independently with a uniform distribution over a circle of circumference 1. If $F_{\ell}(\cdot)$ shows the cdf of the arc length over $[0, 1]$, the distribution of the number of uncovered gaps on the circle, g, is*

$$P(g = m) = \binom{N}{m} \sum_{k=m}^{N} \binom{N-m}{k-m}(-1)^{k-m}\xi_k, \tag{2}$$

where

$$\xi_k = (k-1)! \int_{\sum_{i=1}^{k} u_i = 1} \left[\prod_{i=1}^{k} F_{\ell}(u_i) \right] \left[\sum_{j=1}^{k} \int_{0}^{u_j} F_{\ell}(v) dv \right]^{N-k} d\mathbf{u}. \tag{3}$$

Proof. See [11]. □

To apply Theorem 1 for finding the number of uncovered gaps on $\mathcal{P}$, we first prove the uniformity of the arc distribution over $\mathcal{P}$ in the following lemma.

Lemma 1. *For a symmetric distribution of the sensors over the path, the location of the intersection of the sensors sensing range and $\mathcal{P}$ is uniformly distributed over $\mathcal{P}$.*

Proof. We equivalently show that the center points of the arcs are uniformly distributed over the circle. For this purpose, consider a small element with length $d\ell \rightarrow 0$ on $\mathcal{P}$. Then, we build a sector of the ring based on this length element whose left and right sides pass the left and right ends of the length element (Figure 3). The center point of the arcs, resulted from the intersection of the sensing area of the nodes within the sector and $\mathcal{P}$, falls within $d\ell$. Due to the independence of the sensors distribution from the polar angle, all elements with length $d\ell$ on the circle have the same chance to include an arc center point. Therefore, the distribution of the arc centers, and consequently arc locations, is uniform on $\mathcal{P}$. $\square$

Following Lemma 1, in order to find the distribution of the number of gaps on $\mathcal{P}$, we need $F_\ell(\cdot)$ or in our case $F_a(\cdot)$, the cdf of a_i's. Notice that a_i's are independent and identically distributed (i.i.d) random variables. We find $F_a(\cdot)$ in the appendix for arbitrary distributions of sensor location and sensing range.

As a result of Theorem 1 and Lemma 1, we have the following corollary.

Corollary 1. *The probability of the full path coverage, P_f, is*

$$P_f = P(g = 0)$$

$$= \sum_{k=0}^{N} (-1)^k \binom{N}{k} (k-1)!$$

$$\times \int_{\sum_{i=1}^{k} u_i = 1} \left[\prod_{i=1}^{k} F_a(u_i) \right] \left[\sum_{j=1}^{k} \int_0^{u_j} F_a(v) dv \right]^{N-k} d\mathbf{u}.$$

(4)

Furthermore, one can show that the expected number of gaps on $\mathcal{P}$ is [11]

$$E(g) = N(1 - \mu_a)^{N-1},$$

(5)

where μ_a is the mean of a_i's.

$E(g)$ can be used to find an upper bound on the number of nodes in the network guaranteeing the full path coverage with a given reliability. This is presented below.

Corollary 2. *To guarantee a full path coverage with probability p, the following relation holds*

$$N(1 - \mu_a)^{N-1} \geq 1 - p.$$

(6)

Proof. Recall Markov's inequality for a positive random variable X
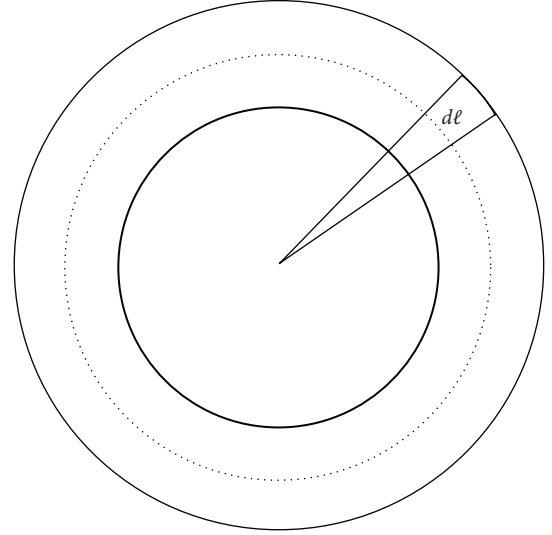
$$P(X \geq b) \leq \frac{E(X)}{b},$$

(7)

FIGURE 3: Distribution of the arc position over the circle.

where $b > 0$. If we let X be the random variable of the number of gaps g, and put $b = 1$, we have

$$P(g \geq 1) \leq E(g).$$

(8)

Combining (5) and (8) results in (6). $\square$

Using (6), it is straightforward to find an upper bound on N guaranteeing a desired level of coverage, p. Later, our simulations show that this bound is in fact very tight.

Another feature of the path coverage that we like to study is the quality of the coverage in terms of the size of the gaps on $\mathcal{P}$. Assume that we like to guarantee detecting any object bigger than a particular size, say t. To assure detecting such objects, all of the gaps have to be smaller than t. Hence, we like to find the probability of having no gaps larger than t, $P(l_g < t)$, where l_g is the length of the largest gap on $\mathcal{P}$.

Corollary 3. *The probability of having no gaps larger than t is*

$$P(l_g < t) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} (k-1)!$$

$$\times \int_{\sum_{i=1}^{k} u_i = 1} \left[\prod_{i=1}^{k} F_a(u_i - t) \right]$$

$$\times \left[\sum_{j=1}^{k} \int_0^{u_j} F_a(v - t) dv \right]^{N-k} d\mathbf{u}.$$

(9)

Proof. Consider a realization of the random placement of arcs on the path. Now, one can consider a scenario where the length of each arc is increased by t. If there exists a gap smaller than t in the first scenario, this gap will be covered in the second scenario since the arcs are t longer. On the other hand, a gap with any size in the second scenario will be a gap with length more than t in the first scenario. Notice

that the above discussion is valid for any realization of the network. Thus, instead of investigating the probability of having no gaps longer than t in the first scenario, we look for the probability of the full coverage in the second scenario. Denoting the length of the arcs in the second scenario by a_i', one can think of them as being drawn from the distribution $f_{a'}(x) = f_a(x - t)$ or equivalently

$$F_{a'}(x) = F_a(x - t). \tag{10}$$

This completes the proof. $\square$

Using the same approach taken for finding the upper bound on N in (6), one can derive an upper bound on the number of nodes to guarantee having all gaps smaller than t.

We also like to investigate $\mathcal{C}$, that is, the portion of $\mathcal{P}$ which is covered by the nodes. To find $\mathcal{C}$, we first reorder the arcs based on their starting points, l_i's. Thus $l_1 < l_2 < l_3 < \cdots < l_N$. Now, we divide $\mathcal{P}$ to arcs b_i, where b_i is an arc starting from l_i and ending at l_{i+1}. Finally, b_N starts from l_N and ends at l_1. Since we have N random arcs intersecting with $\mathcal{P}$, there exist N of such spacings on the circle. These N spacings may or may not be covered by the network. Adding the covered parts of the path together, we have

$$\mathcal{C} = \sum_{i=1}^{N} \min(y_i, x_i) = \sum_{i=1}^{N} z_i, \tag{11}$$

where $z_i = \min(y_i, x_i)$. Also, y_i's denotes the length of b_i and

$$x_i = \max\left(a_i, a_{i-1} - y_{i-1}, \ldots, a_{i-N+1} - \sum_{j=1}^{N-1} y_{i-j}\right). \tag{12}$$

Notice that in (12) we assume rotational indices for a_{i-j}'s. It means that if $i - j < 1$ we replace the index with $N - i + j$. In (12), x_i is the length of the connected part of $\mathcal{C}$ starting from l_i and continuing clockwise. When $y_i \leq x_i$, the whole spacing y_i is covered and $\min(\cdot, \cdot)$ function should return y_i. When $x_i < y_i$, a portion of y_i remains uncovered and there exists a gap at the right side of y_i. Thus, $\min(\cdot, \cdot)$ function returns x_i. It is noteworthy that because of the problem symmetry, z_i's are identically distributed random variables. Thus, we use a single random variable z to refer to them.

The distribution of $\mathcal{C}$ can be well approximated by a Gaussian distribution using central limit theorem (CLT) where the mean value of $\mathcal{C}$, $\mu_{\mathcal{C}}$, is $\mu_{\mathcal{C}} = N\mu_z$. Here, μ_z denotes the mean value of z. Also, $\sigma_{\mathcal{C}}^2 = N\sigma_z^2$ where $\sigma_{\mathcal{C}}^2$ and σ_z^2 represent the variance of $\mathcal{C}$ and z, respectively. In reality, one can safely simplify (12) to

$$x_i = \max(a_i, a_{i-1} - y_{i-1}). \tag{13}$$

This is because a_i's are i.i.d. and thus it is very unlikely that, for example, $a_{i-2} - y_{i-1} - y_{i-2} > a_i$.

3.2. Approximate Analysis. In the following, we present an approximate analysis simplifying our path coverage study. The idea of this approximate analysis is to consider a model where a set of fixed-length arcs are spread randomly over $\mathcal{P}$

instead of using the actual random-sized arcs. The length of these fixed arcs is equal to the mean value of the random-sized arcs in the original case. We denote the mean value of these random arcs with μ_a. In this case, it can be shown that the number of uncovered gaps on $\mathcal{P}$ is distributed as follows [12]:

$$P(g = m) = \binom{N}{m} \sum_{k=m}^{N} (-1)^{k-m} \binom{N - m}{k - m} (1 - k\mu_a)_+^{N-1}, \tag{14}$$

where $(x)_+ = \max(x, 0)$. The same technique as before is applicable to find the probability of having no gaps larger than t on $\mathcal{P}$. For this purpose, we just need to use $\mu_a + t$ instead of μ_a in (14). In addition, the distribution of $\mathcal{C}$ can be derived when the arc size is fixed [12]. In this case, we have

$$F_{\mathcal{C}}(x) = \sum_{j=1}^{N} \sum_{k=0}^{j-1} (-1)^{j+k-1} \binom{N}{j} \binom{N - 1}{k} \binom{j - 1}{k} (1 - x)^k$$
$$\times (x - j\mu_a)_+^{N-1-k}, \tag{15}$$

where $F_{\mathcal{C}}(x)$ is the cdf of $\mathcal{C}$.

One can also calculate the expected value of $\mathcal{C}$, $\mu_{\mathcal{C}}$. To this aim, we first consider the uncovered part of the path, $\mathcal{V}$, and find its expected value, called $\mu_{\mathcal{V}}$. Then $\mu_{\mathcal{C}}$ can be found using the fact $\mu_{\mathcal{C}} = 1 - \mu_{\mathcal{V}}$.

An arbitrary point x on $\mathcal{P}$ remains uncovered when there is no a_i covering it. This is equivalent to having none of l_i's within an arc with length μ_a whose right end point is x. There are N sensors in the network, hence, the probability of having x uncovered, $\mu_{\mathcal{V}}$, is

$$\mu_{\mathcal{V}} = (1 - \mu_a)^N. \tag{16}$$

Consequently,

$$\mu_{\mathcal{C}} = 1 - (1 - \mu_a)^N. \tag{17}$$

3.3. Some Remarks

Remark 1. Our path coverage analysis is applicable to any closed path, for example, ellipse, with finite length when the location of the path segment covered by an arbitrary sensor is uniformly distributed over the path. For this purpose, we just need to have the distribution of the intersection of sensors sensing range and the path. Also, the analysis is applicable to linear path coverage. In fact, the problem of covering a circle with random arcs can be transformed to the problem of covering a piece of line, say the interval $[0, b]$, with random intervals. In this case, sensors are deployed randomly over a strip surrounding the linear path. It is notable that in the linear case, Torus convention [7] is applied. In Torus convention, it is assumed that if a part of the random interval goes out of the line segment, it comes in from the other side of the line piece. However, when the length of the random intervals is small compared to the line piece, one can remove the Torus convention and the analysis remains quite accurate.

Remark 2. In many WSNs, the number of active sensors in the network changes with time. This can be due to, for example, sleep scheduling or death of some nodes. Since our analysis is provided for arbitrary N, it can accommodate such situations, simply by replacing N with $N(t)$ in relevant equations. Consequently, the coverage can be studied as a function of time.

4. Simulation

In this section, we demonstrate the accuracy of our analysis via computer simulations. We have inspected two scenarios for the sensors sensing range. In the first scenario, we assume a network with N sensors all having a fixed sensing range equal to r. The sensors are uniformly deployed inside a ring around the circular path, where $\mathcal{P}$ has unit circumference. In the second scenario, the sensors sensing range is also uniformly distributed between 0 and r_{max}. A zero sensing range can represent a dead sensor.

We evaluate random properties such as the full coverage, number of uncovered gaps, tightness of the bound presented in (6), the intruder detectability, and the portion of the covered path using simulation, and compare the results with our theoretical analysis.

4.1. Uncovered Gaps. Probability density of the number of uncovered gaps on the path, $P(g = m)$, was derived in Section 3.1. Figure 4 shows the probability mass function (pmf) of the number of uncovered gaps via simulation for $N = 30$. Here, we have assumed that the sensing range of all sensor nodes is fixed and is equal to 0.06. The theoretical results using (2) have also been sketched for comparison. It can be concluded that the formulation derived in Section 3.1 quite accurately describes the pmf of the number of uncovered gaps on the path. The third curve in Figure 4 is the result of approximation analysis in Section 3.2. Parameter μ_a in (14) is set to be the expected value of random variable a, derived in the appendix.

It is clear from Figure 4 that the results from the approximate analysis are fairly close to the exact analysis and the simulations. Due to the complexity of the evaluation of exact analysis, we compare the rest of our simulation results with the approximate analysis presented in Section 3.2 to characterize the coverage properties of the network.

In the case of fixed sensing range, as the width of the ring becomes smaller, the variance of a decreases and the arc lengths become closer to μ_a, making the approximate analysis more accurate. To study the worst case, in our analysis, we assume the ring width w is equal to $2r$. Notice that any node outside this ring does not contribute to the path coverage. For random sensing range, we choose $w = r_{max}/2$. Notice that since $r_1 = 0$, there will be nodes in the ring that will not contribute to the path coverage.

Figures 5 and 6 demonstrate the probability of full coverage versus number of sensors deployed in the region. Figure 5 shows the results for the fixed sensing range scenario. P_f is estimated through simulation for different values of sensing range, $r = 0.05, 0.02, 0.01$, and is compared
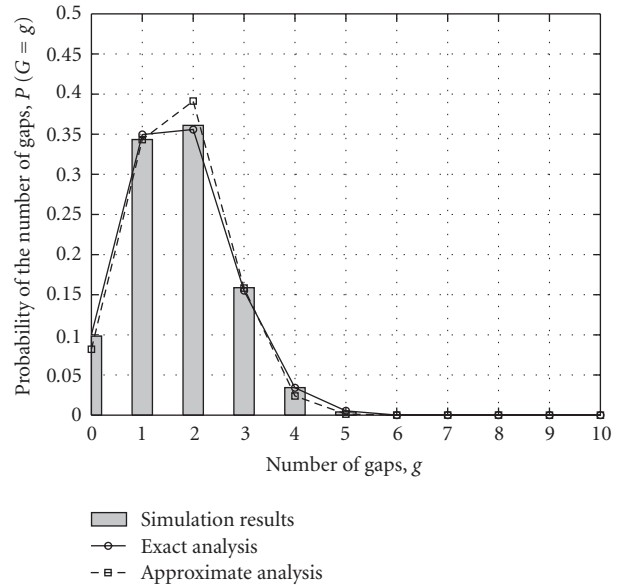

Figure 4: Probability of the number of uncovered gaps when $r = 0.06$ and $N = 30$.

with the theoretical results using (14). As seen from the curves, our theoretical formulation can effectively predict probability of full path coverage. Figure 6 represents the results of variable sensing range scenario. The sensing range of sensor nodes is randomly selected from a uniform distribution between 0 and r_{max}. We have run the simulation based on three uniform distributions, $r_{max} = 0.06, 0.04, 0.02$, and compared with the theoretical results. For theoretical calculations, we have computed the average arc length for the case of random sensing range using (A.22) in the appendix, and then substituted the resulting μ_a into the approximate formula (14). From Figure 6, we can see that the theoretical analysis in Section 3.2 can closely describe the probability of full coverage for random sensing range scenario.

We have also tested our analysis for full coverage of a straight line segment instead of a circle. Figure 7 depicts the coverage of a straight line segment of length 1 when sensing range is fixed and is equal to 0.05. The solid line is the result of Poisson assumption in [2] and the dashed line is the result of our formulation. It can be seen that specially for smaller number of sensor nodes, the Boolean model is not well applicable to describe the coverage of small networks.

The expected number of uncovered gaps, $E(g)$, after deploying N sensors in the ring is given by (5). In Figure 8, $E(g)$ has been calculated versus N for three values of fixed sensing range, $r = 0.05, 0.02, 0.01$, using simulation as well as the analysis. The expected number of gaps for variable sensing ranges is shown in Figure 9. The sensors sensing range has been taken from the three uniform distributions used previously.

In Section 3.1, we used Markov's inequality to find a relation between the number of nodes and probability of full coverage over the path, presented in (6). The smallest number of nodes that satisfies (6) can efficiently be found by conducting a simple binary search. We denote the value
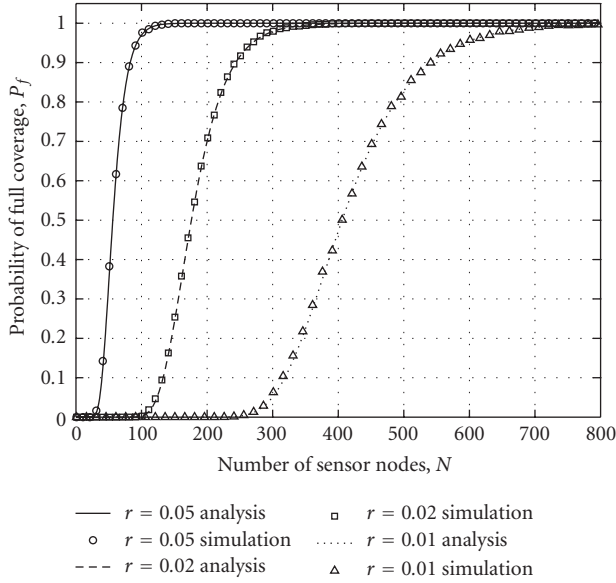
FIGURE 5: Probability of full coverage, that is, $g = 0$, versus the number of sensor nodes for fixed sensing range.
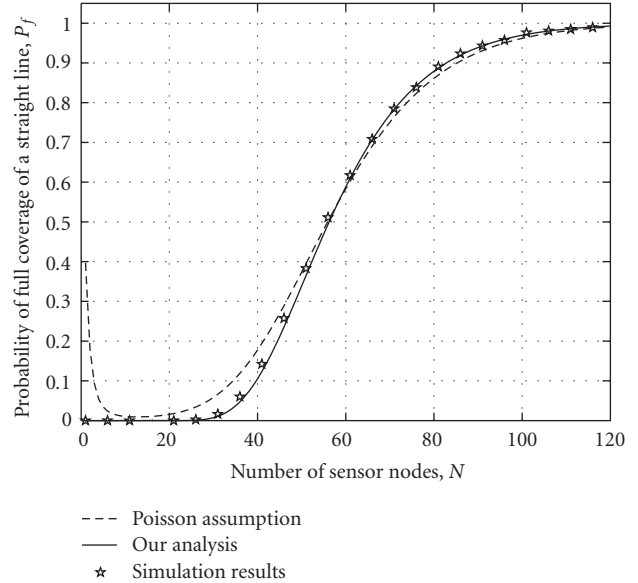

FIGURE 7: Probability of full coverage of a straight linear path of length 1 when $r = 0.05$.
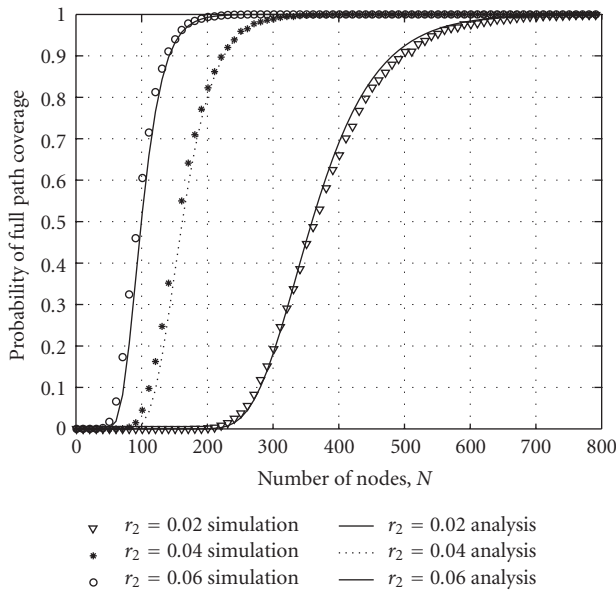

FIGURE 6: Probability of full coverage, that is, $g = 0$, versus the number of sensor nodes for random sensing range.
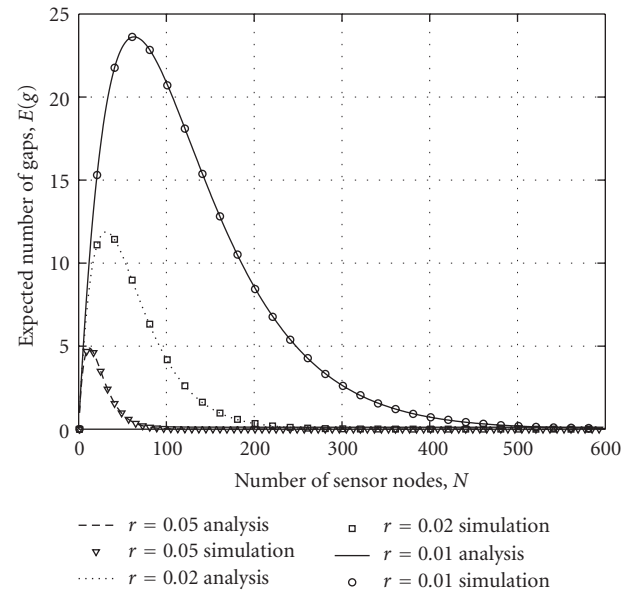

FIGURE 8: Expected value of the number of gaps versus number of sensor nodes for fixed sensing range.

TABLE 1: Upper bound on the number of sensors for guaranteeing full coverage with probability $P = .8$.

	Inequality (6)	Simulation
$r = 0.05$	73	72
$r = 0.02$	220	216
$r = 0.01$	494	489

of N found via search by N^*. Table 1 shows the value of N^* calculated for probability of full coverage being equal to .8. The results found by inequality (6) and simulation are shown for comparison. For probability of full coverage closer to one (the region of interest in practice), N^* gets even closer to the value of N satisfying the desired reliability found via simulation. For example, for probability of full coverage equal to .95, we found $N^* = N$ for various values of r. It can be inferred that (6) provides a tight upper bound on the number of nodes needed for full path coverage.

4.2. Detectability of the Object. As discussed in Section 3.1, in real applications of sensor networks, we might not be interested in the full coverage of a path, yet we need to make
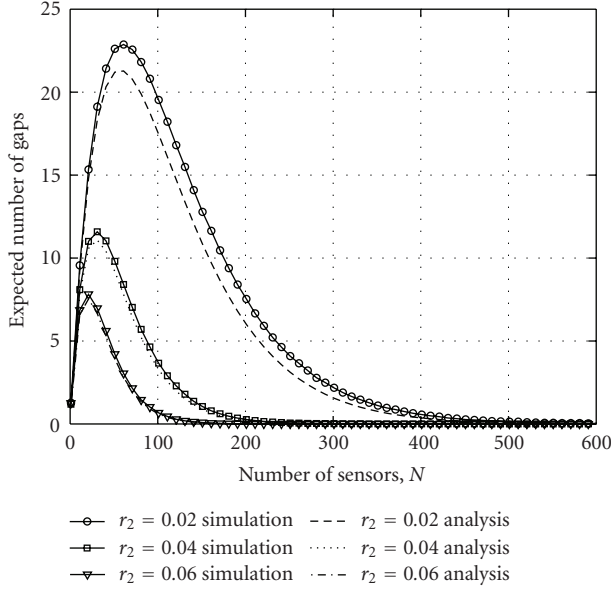
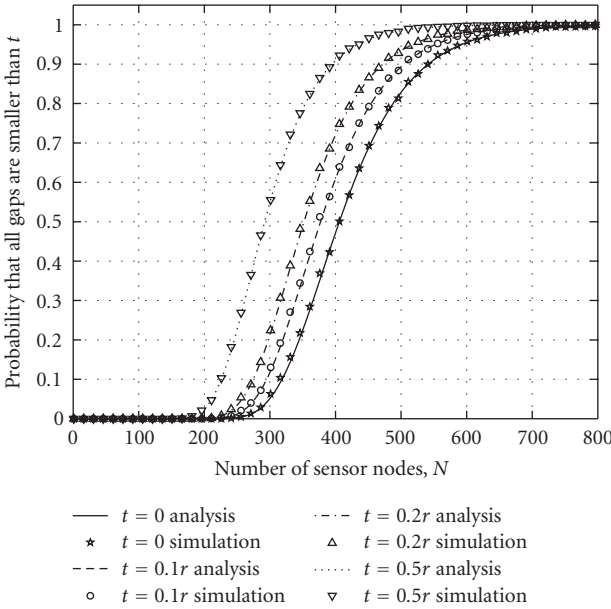FIGURE 9: Expected value of the number of gaps versus number of sensor nodes for random sensing range.

FIGURE 11: The cdf of the covered part of the path when $N = 30$.

FIGURE 10: Probability of having no gaps larger than t when $r = 0.01$.
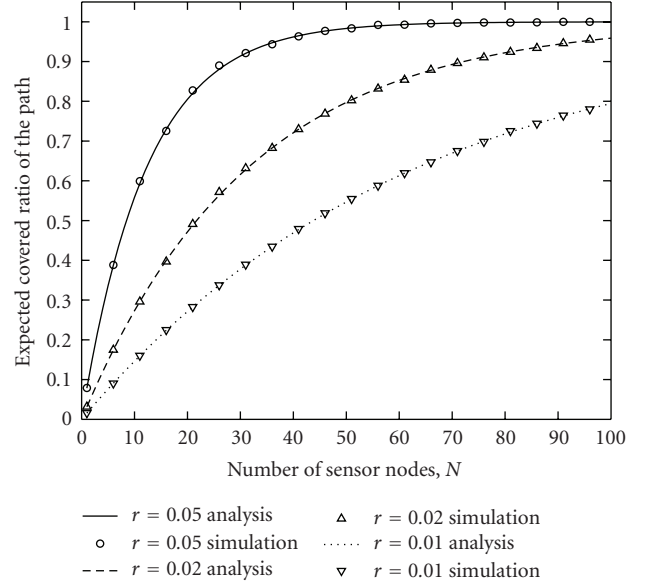
FIGURE 12: Expected size of the covered portion of the path versus N.

sure that there are no uncovered gaps on the path, larger than a certain maximum length t. The probability of this kind of coverage, $P(l_g < t)$, was given by (9). We use simulation to find $P(l_g < t)$ for values of t equal to $r/2$, $r/5$, $r/10$, when $r = 0.01$. Again, comparing simulation results with theoretical ones in Figure 10 verifies our formulation. Our study on the size of the gaps is useful for decreasing the cost of the network implementation. In fact, if we know the size of the intruders, instead of providing a full path coverage, we can design the network with less number of nodes to have all gaps smaller than t.
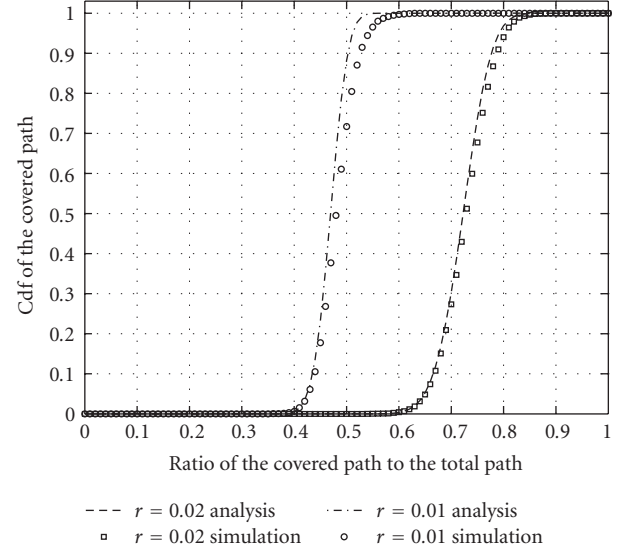
4.3. Covered Part of the Path. The covered portion of the path, $\mathcal{C}$, is another important metric for path coverage in a WSN. Indeed, $\mathcal{C}$ is a random variable whose cdf is approximated in Section 3.2. Figure 11 shows the cdf of $\mathcal{C}$, for $N = 30$ and $r = 0.02, 0.01$. As it can be seen, our path coverage analysis is more accurate for larger values of r.

The formulation for expected covered part, $\mu_{\mathcal{C}}$, is derived in Section 3.2. Figure 12 shows simulation and theoretical results for $\mu_{\mathcal{C}}$ versus N, when $r = 0.05, 0.02$, and 0.01.

5. Conclusion

In this paper, we studied the path coverage of a random WSN when neither the area size nor the number of network nodes were infinite. Hence, the widely used Boolean model was no longer valid. Moreover, due to the randomness of the sensors placement over the area, network coverage was nondeterministic. Thus, a probabilistic solution was taken for determining the network coverage features. Our analysis considered the number of gaps, probability of full path coverage, probability of having all uncovered gaps smaller than a specific size, and the cdf of the covered length of the path. All these characteristics were found as a function of the number of sensors N. We also proposed a tight upper bound on required N for full coverage. Through computer simulations, we verified the accuracy of our approach. Since our study was performed for finite N, using our results on various features of path coverage, one can find the necessary number of sensors for a certain quality of coverage.

Appendix

In the following, we find the cdf of the intersection between the sensing area of the sensors and $\mathcal{P}$, called $F_a(x)$. First, we study the situation where sensors have a fixed sensing range r and they are uniformly distributed over the ring. Then, we investigate the general case where sensors can have a random sensing range varying from r_1 to r_2 and have any symmetric distribution over the ring.

Let us first discuss the case where the sensors have a fixed sensing range. Figure 13 shows the ring-shaped network containing $\mathcal{P}$. As mentioned previously, the circumference of $\mathcal{P}$ is 1, hence, the radius of $\mathcal{P}$ is $R = 1/2\pi$. It is also assumed that the ring width is $2w$ and $w \leq r$, where r is the sensing radius of the sensors. Notice that d in Figure 13 shows the distance of a sensor from the center of the ring. Since the sensors are uniformly distributed over the area, it can be easily shown that the cdf of d, $F_d(x)$, is as follows:

$$F_d(x) = \begin{cases} 0, & x < R - w, \\ \dfrac{x^2 - (R - w)^2}{4wR}, & R - w \leq x \leq R + w, \\ 1, & R + w < x. \end{cases} \quad \text{(A.1)}$$

We use $F_d(x)$ to derive $F_a(x)$.

In Figure 13, the intersection of the sensing area of an arbitrary sensor with $\mathcal{P}$ is denoted by a. By forming a triangle whose vertices are the center of $\mathcal{P}$, sensor location, and one of the points where the sensing circle of the sensor meets $\mathcal{P}$, one can write

$$r^2 = R^2 + d^2 - 2Rd\cos(\theta). \quad \text{(A.2)}$$

On the other hand, we have

$$a = 2R\theta. \quad \text{(A.3)}$$

Replacing θ with $a/2R$ in (A.2) results in

$$d^2 - 2Rd\cos\left(\frac{a}{2R}\right) + (R^2 - r^2) = 0. \quad \text{(A.4)}$$

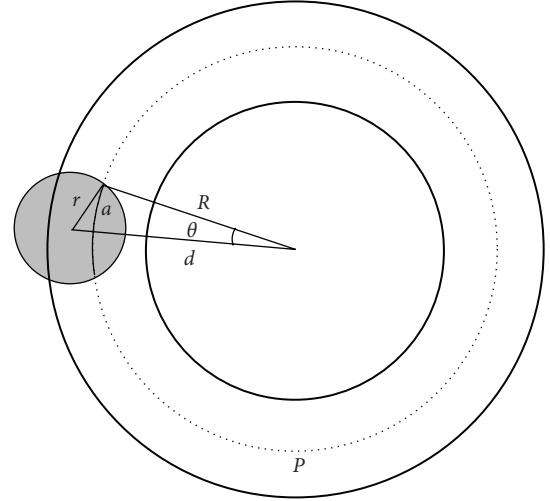

FIGURE 13: Covered part of the path by a single sensor.

Solving (A.4), we have

$$d = R\cos\left(\frac{a}{2R}\right) \pm \sqrt{r^2 - R^2\sin^2\left(\frac{a}{2R}\right)}. \quad \text{(A.5)}$$

Equivalently,

$$a = 2R\cos^{-1}\left(\frac{R^2 + d^2 - r^2}{2Rd}\right). \quad \text{(A.6)}$$

Now having the cdf of d and using the relation between d and a in (A.5) and (A.6), we will derive $F_a(x)$. To this end, one can state

$$
\begin{aligned}
P(a \leq x) &= P\left(2R\cos^{-1}\left(\frac{R^2 + d^2 - r^2}{2Rd}\right) \leq a\right) \\
&= P\left(d \leq R\cos\left(\frac{a}{2R}\right) - \sqrt{r^2 - R^2\sin^2\left(\frac{a}{2R}\right)}\right) \\
&\quad + P\left(d \leq R\cos\left(\frac{a}{2R}\right) + \sqrt{r^2 - R^2\sin^2\left(\frac{a}{2R}\right)}\right).
\end{aligned}
\quad \text{(A.7)}
$$

Thus,

$$
\begin{aligned}
F_a(x) &= 1 + F_d\left(R\cos\left(\frac{x}{2R}\right) - \sqrt{r^2 - R^2\sin^2\left(\frac{x}{2R}\right)}\right) \\
&\quad - F_d\left(R\cos\left(\frac{x}{2R}\right) + \sqrt{r^2 - R^2\sin^2\left(\frac{x}{2R}\right)}\right).
\end{aligned}
\quad \text{(A.8)}
$$

Replacing $F_d(x)$ in (A.8) using (A.1), we obtain

$$F_a(x) = 1 - \frac{\cos(x/2R)\sqrt{r^2 - R^2\sin^2(x/2R)}}{w}, \quad \text{(A.9)}$$

where

$$c \leq x \leq 2R \sin^{-1}\left(\frac{r}{R}\right),$$

$$c = 2R \cos^{-1}\left(\sqrt{\frac{R^2 - r^2 + \sqrt{(R^2 - r^2)^2 + 4w^2 R^2}}{2R^2}}\right).$$
$$(A.10)$$

Moreover, $F_a(x) = 0$ when $x < c$ and $F_a(x) = 1$ when $x > 2R \sin^{-1}(r/R)$. We use (A.9) for our exact analysis in order to characterize the path coverage features of the network. Notice that when r is small, $F_a(x)$ can be approximated as follows:

$$F_a(x) = 1 - \frac{\sqrt{r^2 - (x/2)^2}}{w}.$$
$$(A.11)$$

In addition to the cdf of the arc length, we use the mean value of a for our approximate analysis. Recall that for an arbitrary random variable z distributed over $[a, b]$,

$$\mu_z = b - \int_a^b F_z(x)dx,$$
$$(A.12)$$

where μ_z is the mean value of z and $F_z(x)$ is the cdf of z. Using (A.12), μ_a can be found as follows:

$$\mu_a$$

$$= \int_c^{2R \sin^{-1}(r/R)} F_a(x)dx$$

$$= \left[x - \frac{\sqrt{r^2 - R^2 \sin^2(x/2R)}}{w} \right.$$

$$\left. - \frac{r^2 \tan^{-1}\left(R \sin(x/2R)/\sqrt{r^2 - R^2 \sin^2(x/2R)}\right)}{w} \right]_c^{2R \sin^{-1}(r/R)}.$$
$$(A.13)$$

Notice that when $w = r$, $\mu_a = \pi r/2$.

Now assume that both sensing range and sensor location are random and we like to find $F_a(x)$. Sensing range of the sensors, r, varies over $[r_1, r_2]$ with probability density function (pdf) $f_r(x)$. Also, $R - w \leq d \leq R + w$ such that $w \leq r_2$, because sensors located farther than r_2 from the path do not contribute in the path coverage. It is noteworthy that $a \in [0, a_1]$ where

$$a_1 = 2R\cos^{-1}\left(\frac{2R^2 - r_2^2}{2Rd}\right).$$
$$(A.14)$$

This can simply be justified using (A.6).

To find $F_a(x)$, we partition the problem to two separate cases. In the first case, sensing area of the sensor does not intersect with $\mathcal{P}$, that is, $a = 0$. This happens when $d + r \leq$

R or $d - r \geq R$. If $w < r_1$, this never happens and sensing area of the sensor always intersects with $\mathcal{P}$ and consequently $F_a(0) = 0$. If $w \geq r_1$, we have

$$F_a(0) = P(d + r < R) + P(d - r > R).$$
$$(A.15)$$

To evaluate two terms in the right side of the above equation, we use the joint distribution of r and d, $f_{r,d}(x, y)$. Notice that in the case where sensors sensing range is independent from their location, $f_{r,d}(x, y) = f_r(x) \cdot f_d(y)$, where $f_d(y)$ is the pdf of d over $[R - w, R + w]$. To evaluate $P(d + r < R)$ and $P(d - r > R)$, we simply have to integrate from $f_{r,d}(x, y)$ over the area where $d + r \leq R$ or $d - r \geq R$. It can be shown that

$$F_a(0) = \int_{r_1}^{w} \int_{R-w}^{R-r} f_{r,d}(x, y)dy\,dx + \int_{r_1}^{w} \int_{R+r}^{R+w} f_{r,d}(x, y)dy\,dx.$$
$$(A.16)$$

Notice that $F_a(0) > 0$ states that the pdf of a, $f_a(x)$, has a Dirac delta function at $x = 0$.

When sensing area of a sensor intersects with path, $a > 0$. To find $F_a(x)$ in this case, we first find $f_a(x)$. For this purpose, we apply Jacobian transformation to derive $f_{a,d}(x, y)$, the joint distribution of a and d, from $f_{r,d}(x, y)$. Using (A.6) and Jacobian transformation, one can show that

$$f_{a,d}(x, y) = \frac{y\sqrt{1 - \cos^2(x/2R)}}{2z} f_{r,d}(z, y),$$
$$(A.17)$$

where

$$z = \sqrt{R^2 + y^2 - 2Ry \cos\left(\frac{x}{2R}\right)}.$$
$$(A.18)$$

Having $f_{a,d}(x, y)$, $f_a(x)$ is found by integration over d. To integrate over d, the region of integration has to be determined carefully. For any arbitrary value of $a \in (0, a_1]$, there exist an infinite number of pairs (r, d) satisfying (A.6); however, to guarantee an intersection between the sensing range of the sensor and $\mathcal{P}$, d should fall within $[d_1, d_2]$ where

$$d_1 = R \cos\left(\frac{a}{2R}\right) - \sqrt{r_2 - R^2 \sin^2\left(\frac{a}{2R}\right)},$$

$$d_2 = R \cos\left(\frac{a}{2R}\right) + \sqrt{r_2 - R^2 \sin^2\left(\frac{a}{2R}\right)}.$$
$$(A.19)$$

In fact, d_1 and d_2 are the desired integral bounds. Thus,

$$f_a(x) = \int_{d_1}^{d_2} f_{a,d}(x, y)dy \quad x > 0.$$
$$(A.20)$$

Consequently,

$$F_a(x) = \int_{0^-}^{x} f_a(y)dy \quad x > 0.$$
$$(A.21)$$

The mean value of a, μ_a, used in our approximate analysis, is also derived as follows:

$$\mu_a = \int_0^{a_1} x f_a(x) dx$$

$$= 0 \times F_a(0) + \int_{0^-}^{a_1} x f_a(x) dx \qquad (A.22)$$

$$= \int_{0^-}^{a_1} x f_a(x) dx.$$

Acknowledgment

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