# Exact Symbol Error Probability of Cross-QAM in AWGN and Fading Channels

| ArticleInfo | | |
|---|---|---|
| ArticleID | : | 2059 |
| ArticleDOI | : | 10.1155/2010/917954 |
| ArticleCitationID | : | 917954 |
| ArticleSequenceNumber | : | 272 |
| ArticleCategory | : | Research Article |
| ArticleFirstPage | : | 1 |
| ArticleLastPage | : | 1 |
| ArticleHistory | : | RegistrationDate : 2010–4–28<br>Received : 2010–4–28<br>Revised : 2010–9–24<br>Accepted : 2010–11–21<br>OnlineDate : 2010–11–30 |
| ArticleCopyright | : | Xi-chun Zhang et al.2010<br>This article is published under license to BioMed Central Ltd. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. |
| ArticleGrants | : | |
| ArticleContext | : | 136382010201011 |

Xi-chun Zhang,[Aff1]
**Email**: eexichunzh@gmail.com

Hua Yu,[Aff1]
Corresponding Affiliation: Aff1
**Email**: yuhua@scut.edu.cn

Gang Wei,[Aff1]
**Email**: eegwei@gmail.com

Aff1 Guangdong Provincial Key Laboratory of Short-Range Wireless Detection and Communication, School of Electronic and Information Engineering, South China University of Technology, Guangzhou 510640, China

## Abstract

The exact symbol error probability (SEP) performance of $M$-ary cross quadrature amplitude modulation (QAM) in additive white Gaussian noise (AWGN) channel and fading channels, including Rayleigh, Nakagami-m, Rice, and Nakagami-q (Hoyt) channels, is analyzed. The obtained closed-form SEP expressions contain a finite (in proportion to $\sqrt{M}$) sum of single integrals with finite limits and an integrand composed of elementary (exponential, trigonometric, and/or power) functions, thus readily enabling numerical evaluation. Particularly, Gaussian $Q$-function is a special case of these integrals and is included in the SEP expressions. Simple and very precise approximations, which contain only Gaussian $Q$-function for AWGN channel and contain three terms of the single integrals mentioned above for fading channels, respectively, are also given. The analytical expressions show excellent agreement with the simulation results, and numerical evaluation with the proposed expressions reveals that cross QAM can obtain at least 1.1 dB gain compared to rectangular QAM when SEP < 0.3 in all the considered channels.

# 1. Introduction

Quadrature amplitude modulation (QAM) has been widely used in digital communication systems due to its high bandwidth efficiency. When the number of bits per symbol is even, transmission can be implemented easily by using square QAM. However, if there is a requirement for the transmission of an odd number of bits per symbol, the rectangular QAM is not a good choice in terms of power efficiency. The issue was overcome by Smith who proposed cross-QAM constellation which is obtained from a square constellation by removing some outer point in each corner and is given the shape of a cross [1]. Smith shows that both the peak and average power can be reduced by using a cross-QAM constellation, and there is at least a 1-dB gain in the average signal-to-noise ratio.

Recently, cross-QAM has been found to be useful in adaptive modulation schemes wherein the constellation size is adjusted depending on the channel quality [2–6]. As the channel quality improves, the constellation size is expected to be increased by incrementing $2^m$ to $2^{m+1}$. If one were to use just square QAM, the increments should be from $2^m$ to $2^{m+2}$ (for instance, we need to go from 16 to 64 to 256-QAM …). Using cross-QAM, however, the increment is smoother (16-QAM to cross 32-QAM to 64-QAM …). The steps between consecutive squared constellations are too big, especially for small constellations [7]. An intermediate step (corresponding to odd powers of 2) will make the system to work with more granularity obtaining greater coverage for a determined data rate [2]. As a result, cross-QAMs have been adopted in many practical systems. For example, cross-QAMs with constellations from 5 bits to 15 bits have been used in ADSL and VDSL [8, 9], and cross 32-QAM and cross 128-QAM are adopted in DVB-C [10]. On the other hand, cross-QAMs have special application in blind equalization [11–14].

Despite the immense importance of cross-QAM, the implementation and the calculation of the average symbol error probability (SEP) of cross-QAM are more complicated compared to that of square and rectangular QAMs since the inphase and quadrature components of cross-QAMs cannot be demodulated independently. So, the calculation of SEP of cross-QAM cannot be reduced to a one-dimensional problem by using the independence of the inphase and quadrature components, as can be done for square QAM and rectangular QAM. It can be found the upper bounds of the SEP of cross-QAM in [1] and [15, Chapter 5]. Recently, [16] derives the bit error probability (BEP) of cross-QAM constellations with Smith-style Gray coding in additive white Gaussian noise (AWGN) and Rayleigh flat fading. However, the closed-form solutions to BEP for AWGN and Raleigh flat fading channels in [16] are very complicated, not to mention that [16] does not consider the SEP of cross-QAM and that it is not straightforward to extend the result of [16] to SEP calculation. More recently, an exact closed-form SEP expression for 32 cross-QAMs in AWGN has been derived in the form of a finite sum of Gaussian $Q$-functions [17]. Note, however, that the decision regions of the corner points for other cross-QAM modulation are more complicated than 32-cross-QAM. In [18], an infinite double series of the products of Gaussian $Q$-functions for the SEP of 128-cross-QAM and 512-cross-QAM in AWGN is derived. The difficulty arises when one tries to find the closed-form exact SEP expression of arbitrary $M$-ary cross-QAM in AWGN, which involves the calculation of the two-dimensional joint Gaussian $Q$-function.

In fading channels, on the other hand, the signal-to-noise ratio (SNR) of the received symbol becomes a random variable. As such, the usual method to find the average SEP in fading channels is to first get the conditional SEP as if in an AWGN channel (conditioned on SNR or channel realization) then average the conditional SEP over SNR or channel realization. This approach will, in many cases, result in an integral, usually being two-dimensional integral even three-dimensional integral, that cannot be integrated into a closed form and hence must be integrated numerically. By using the "preaveraging" technique [19], opposed to the customary and widely adopted "postaveraging" technique, the exact SEP of cross-QAM in Rayleigh fading channels has been derived in [20], where the closed-form SEP expression obtained contains only elemental functions (trigonometric). However, the "preaveraging" technique involves tedious calculations again, and it would only apply for Nakagami-$m$ channels with integer parameter $m$, not to mention whether it can be extended to encompass Rice and Hoyt fading. Perhaps, the best method to get SEP of digital modulation signaling over fading channels is given by Simom and Alouini [21]. They use the moment generating function- (MGF-) based approach to obtain SEP expressions for various digitally modulated signals over fading channels. Their basic technique is to rewrite the Gaussian $Q$-function into a preferred form of an integral with finite integration limits (many SEP expressions contain the Gaussian $Q$-function), so that the final average SEP expression can be numerically computed with more accuracy. In some special cases, the MGF-based method can even lead to an exact closed-form SEP without undone integrals.

In this paper, using the alternate representation of the two-dimensional joint Gaussian $Q$-function and the MGF-based method, the exact SEP expressions of arbitrary $M$-ary cross-QAM in AWGN and fading channels, including Rayleigh, Nakagami-m, Rice, and Nakagami-q (Hoyt) channels, have been derived. The closed-form SEP expressions obtained contain Gaussian $Q$-functions and a finite (in proportion to $\sqrt{M}$) sum of single integrals with finite limits and an integrand composed of elementary (exponential, trigonometric, and power) functions, thus readily enabling numerical evaluation.

The remainder of this paper is organized as follows. First, Section 2 briefly introduces the problem background. Next, Sections 3 and 4 derive the average SEP expressions for cross-QAM in AWGN channels and fading channels, respectively. And then, Section 5 gives the simulation and numerical results. Finally, Section 6 concludes the contributions of this work.

# 2. Background

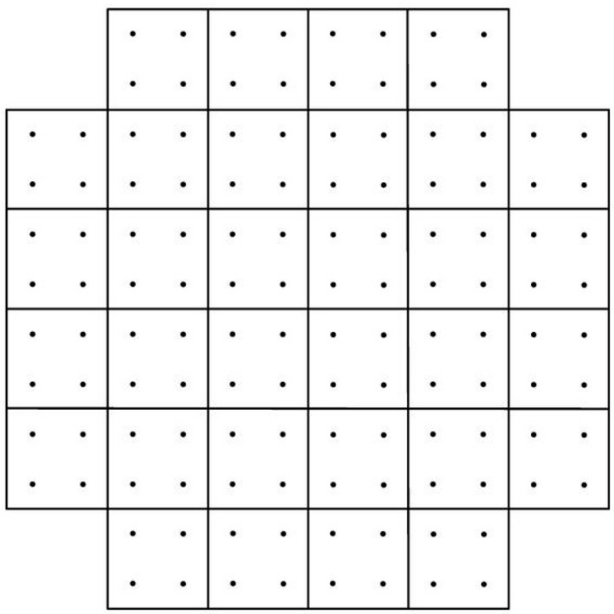
## 2.1. Construction of Cross-QAM

Since we deal with cross-QAMs in this paper, we define $M = 2^{2m+1}$ with $m \geq 2$, that is, $M = 32, 128, 512, 2048, \ldots$. As mentioned in [1], by introducing "block" parameter

$$v \triangleq \sqrt{\frac{M}{32}}, \tag{1}$$

cross-QAM constellation can be constructed by a $6 \times 6$ square block array with the 4 corner blocks deleted, each block with $v^2$ uniform distributed points, as shown in Figure 1.

**Figure 1  Block structure of XQAM.** A $6 \times 6$ square block array with the 4 corner blocks deleted, each block with $v^2$ uniform distributed points. Especially, cross 128-QAM ($v = 2$) is shown here as an example.

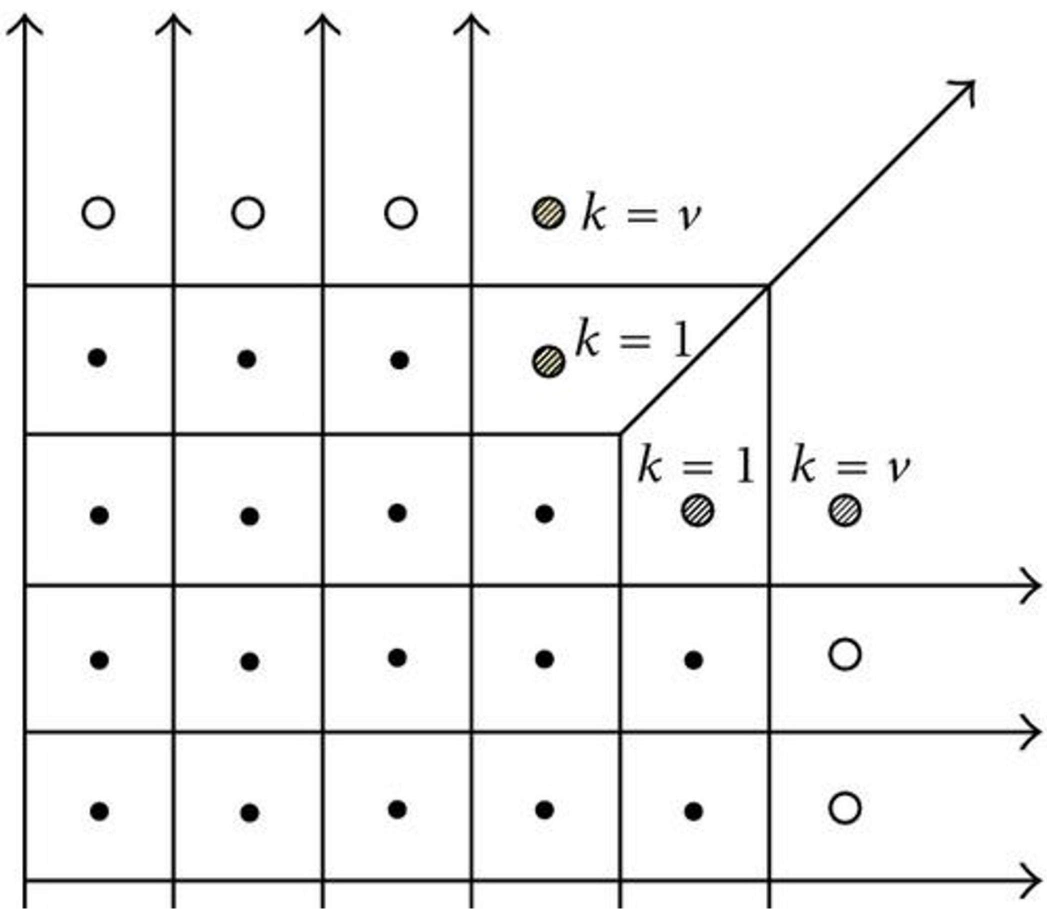


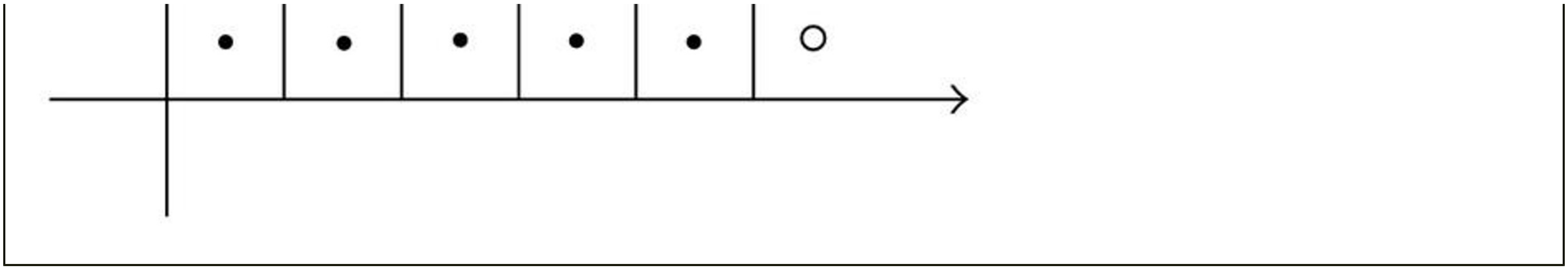
## 2.2. Decision Boundaries for Symbols in Cross-QAM

Since the quadrature components of cross-QAM are not independent as in square QAM, the optimal decision regions are not all rectangular. As shown in Figure 2, where the dots represent signal points while the lines indicate decision boundaries, there are three types of points: interior symbols, edge symbols, and corner symbols. The decision regions of interior symbols and edge symbols are closed square and semi-infinite rectangular, respectively. But that of the corner symbols are slightly complicated and can be represented by a combination of vertical, horizontal, and 45° lines. According to the symmetry, it is enough to consider the symbols in one quadrant of the cross-QAM constellation. The numbers of the three types of points in one quadrant of the constellation are, respectively, given as

$$\begin{aligned} N_{in} &= (2v)^2 + 2(v-1)(2v-1) = 8v^2 - 6v + 2, \\ N_{ed} &= 2(2v-1) = 4v - 2, \\ N_{co} &= 2v. \end{aligned} \tag{2}$$

**Figure 2 Decision boundaries for symbols in cross-QAM.** The black points, hollow points, and slash points represent the interior symbols, edge symbols, and corner symbols, respectively. In particular, only one quadrant of the cross 128-QAM constellation ($v = 2$) is shown here as an example.





## 2.3. System Parameters for $M$-ary Cross-QAM

We will define some parameters in order to make our expressions simpler and compact. Let $d$ denote half of the minimum Euclidean distance between adjacent symbols in the constellation, and let $N_0/2$ denote the two-sided power spectral density of the zero-mean AWGN (i.e., its variance $\sigma^2 = N_0/2$). Especially, the exact SEP expressions will be written in terms of the Gaussian $Q$-function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt, \tag{3}$$

and a well-known integral function related to the alternate representation of one- and two-dimensional joint Gaussian $Q$-functions [21–24]

$$Q_a(x, \phi) = \frac{1}{2\pi} \int_0^{\phi} \exp\left\{-\frac{x^2}{2\sin^2\theta}\right\} d\theta, \quad x \geq 0. \tag{4}$$

In particular, $Q(x) = Q_a(x, \pi) = 2Q_a(x, \pi/2)$ [22, equation $(2)$] and $Q^2(x) = 2Q_a(x, \pi/4)$ [23, equation $(12)$], both for $x \geq 0$.

Note that, in addition to the advantage of having finite integration limits, the form in (4) has the argument $x$ contained in the integrand rather than in the integration limits as is the case in (3), and it also has an integrand that is exponential in the argument $x$, so that it can be numerically evaluated with more accuracy. Moreover, the form in (4) has some interesting implications with regard to simplifying the evaluation of performance results related to communication problems, for example, as seen later to the SEP performance evaluation over fading channels, wherein the argument of the $Q_a$-function is dependent on random system parameters and, thus, requires averaging over the statistics of these parameters.

To simplify the mathematical expressions, in the following derivation the argument $x$ of the above two functions (3) and (4) sometimes will be expressed as a multiple of

$$\delta \triangleq \frac{d}{\sqrt{N_0/2}}, \tag{5}$$

which denotes the normalized least distance (in noise standard deviation) from a signal point to a decision boundary.

Assuming that the signal points are equally probable and according to the symmetry of the constellation, it can be easily shown that the average symbol energy for cross $M$-QAM constellation is given by

$$\varepsilon_{av} = \frac{4}{M}\left\{2 \cdot 3v \sum_{k=1}^{3v} (2k-1)^2 d^2 - 2v \sum_{k=2v+1}^{3v} (2k-1)^2 d^2\right\} = \frac{d^2}{A}, \tag{6}$$

where

$$A \triangleq \frac{48}{31M-32}. \tag{7}$$

Since the symbol's signal-to-noise ratio (SNR) can be written as

$$\gamma = \frac{\varepsilon_{av}}{N_0} = \frac{d^2}{AN_0} = \frac{\delta^2}{2A}, \tag{8}$$

thus,

$$\delta = \sqrt{2A\gamma}. \tag{9}$$

So, we can also leave the SEP expression in terms of $\delta$ as mentioned above.

# 2.4. Overview of SEP Approximations in AWGN

For nonrectangular $M$-ary QAM signal constellations, Proakis has given an obvious upper bound in [15, page 279] as

$$P_{\text{Proakis}}(M) < (M-1)Q\left(\sqrt{\frac{\left(d_{\min}^{(e)}\right)^2}{2N_0}}\right), \tag{10}$$

where $d_{\min}^{(e)}$ is the minimum Euclidean distance between signal points, and $d_{\min}^{(e)} = 2d$ for uniform cross-QAM. This bound may be loose when $M$ is large. In such a case, Proakis suggested replacing $(M-1)$ by $M_n$, the largest number of neighboring points that are at distance $d_{\min}^{(e)}$ from any constellation point. Obviously, for cross-QAM, $M_n = 4$. So, (10) can be reproduced for cross-QAM as

$$P_{\text{Proakis}}(\gamma) < 4Q\left(\frac{d}{\sqrt{N_0/2}}\right) = 4Q\left(\sqrt{2A\gamma}\right). \tag{11}$$

Alternately, Gilbert approximation [1, equation (1)] to the SEP for any $M$-ary QAM can also be used for cross-QAM, and it is given as

$$P_{\text{Gilbert}}(\gamma) \simeq N_n Q\left(\frac{d}{\sqrt{N_0/2}}\right) = N_n Q\left(\sqrt{2A\gamma}\right), \tag{12}$$

where $N_n$ is the average number of nearest neighbors for a symbol in the constellation, and $N_n = 4 - 6/(8v) = 4 - 6/\sqrt{2M}$ for cross-QAM (In [1], the expression for $N_n$ appears erroneously as $N_n = 4 - 5/(8v)$ [16].) when $M \geq 32$. Note that, as $M$ increases, $N_n$ increases and approaches $M_n = 4$. In fact, the principle behind the above two approximate expressions is very intuitive since they can be interpreted as the sum of the probabilities that a given point is mistaken for its neighbors. At the same time, since the sum has recalculate some error regions, both the above two expressions overestimate the actual SEP as shown later.

# 3. Symbol Error Probability in AWGN Channel

While the SEP expressions for the interior symbols and edge symbols can be deduced easily from the SEP expressions for square QAM in [15, pages 265 and 278], the SEP of corner symbols needs to be derived separately.

The SEP of any interior symbol and edge symbol can be written, respectively, as

$$\begin{aligned} P_{\mathrm{in}} &= 1-(1-2Q(\delta))^2 = 4Q(\delta)-4Q^2(\delta), \\ P_{\mathrm{ed}} &= 1-(1-Q(\delta))(1-2Q(\delta)) = 3Q(\delta)-2Q^2(\delta)\,. \end{aligned} \tag{13}$$

For the corner symbols, from the left to right or from the bottom to the top as shown in Figure 2, the SEP of the $k$th point is given by

$$\begin{aligned} P_{\mathrm{co}}^{k} &= 1-\frac{1}{2}(1-2Q(\delta))^2-\int_{-\delta}^{\delta}\left(\frac{1}{2}-Q(2k\delta+x)\right)p(x)\,dx \\ &= 1-\frac{1}{2}(1-2Q(\delta))^2-\frac{1}{2}(1-2Q(\delta))+\Phi(k,\delta;\delta) \\ &= 3Q(\delta)-2Q^2(\delta)+\Phi(k,\delta;\delta), \qquad k=1,2,\ldots,v-1, \\ P_{\mathrm{co}}^{v} &= 1-\frac{1}{2}(1-2Q(\delta))(1-Q(\delta)) \\ &\quad -\int_{-\delta}^{\infty}\left(\frac{1}{2}-Q(2v\delta+x)\right)p(x)\,dx \\ &= \frac{1}{2}(1+3Q(\delta)-2\,Q^2(\delta))-\frac{1}{2}(1-Q(\delta))+\Phi(v,\delta;\infty) \\ &= 2Q(\delta)-Q^2(\delta)+\Phi(v,\delta;\infty), \end{aligned} \tag{14}$$

where $p(x)$ is the probability density function (PDF) of the standard normal distribution and given by

$$p(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \tag{15}$$

$$\Phi(k,\delta;b) \triangleq \int_{-\delta}^{b} Q(2k\delta+x)p(x)\,dx\,. \tag{16}$$

Having derived the exact SEP of all the points, the exact average SEP for the $M$-ary cross-QAM constellation can be written as

$$\begin{aligned} P_{exa}(\delta) &= \frac{4}{M}(N_{in}P_{in} + N_{ed}P_{ed} + 2\textstyle\sum_{k=1}^{v} P_{co}^{k}) \\ &= \frac{4}{M}\{(M-6v)Q(\delta) - (M-12v+2)Q^2(\delta) \\ &\quad +2\Phi(v,\delta;\infty) + 2\textstyle\sum_{k=1}^{v-1}\Phi(k,\delta;\delta)\}\,. \end{aligned} \tag{17}$$

Appendix shows that, with the integral function defined in (4), (17) can be rewritten as

$$\begin{aligned} P_{exa}(\gamma) &= g_1 Q(\sqrt{2A\gamma}) + \frac{4}{M}Q(2\sqrt{A\gamma}) - g_2 Q^2(\sqrt{2A\gamma}) \\ &\quad -\frac{16}{M}\textstyle\sum_{k=1}^{v-1} Q_a(\sqrt{2A\gamma},\alpha_k^+) - \frac{8}{M}\textstyle\sum_{k=1}^{v-1} Q_a(2k\sqrt{A\gamma},\beta_k^+) \\ &\quad +\frac{8}{M}\textstyle\sum_{k=2}^{v} Q_a(2k\sqrt{A\gamma},\beta_k^-), \end{aligned} \tag{18}$$

where

$$\begin{aligned} g_1 &= 4 - \frac{6}{\sqrt{2M}}, \\ g_2 &= 4 - \frac{12}{\sqrt{2M}} + \frac{12}{M}, \\ \alpha_k^+ &= \arctan\left(\frac{1}{2k+1}\right), \quad k = 1,\dots,v-1, \\ \beta_k^{\pm} &= \arctan\left(\frac{k}{k\pm1}\right), \quad k = 1,\dots,v\,. \end{aligned} \tag{19}$$

Although it is difficult to express the function $Q_a(x,\phi)$ in a closed form without integration, it is a one-dimensional integral with finite limits, and its integrand only composes of elementary (exponential and trigonometric) functions, while the function $\Phi(k,\delta;b)$ actually is a two-dimensional integral of exponential function with infinite limits according to (A.2) in Appendix. With (4), in fact, $Q_a(x,\phi)$ can be easily and accurately evaluated numerically (see, e.g., using MATLAB). Note that the numerical calculation of (4) is much simpler and more precise than that of the infinite double series of the products of Gaussian $Q$-functions of equation (3) in [18].

Note that, when $M = 32$, only the first three terms of the right side of (18) are retained, while the last three terms disappear since $v = 1$. So, the exact SEP expression of 32-ary cross-QAM is

$$P_{exa}(\gamma) = \frac{1}{8}[26Q(\sqrt{2A\gamma}) + Q(2\sqrt{A\gamma}) - 23Q^2(\sqrt{2A\gamma})], \tag{20}$$

which is the same as equation (5) in [17].

If ignoring the last three terms and only retaining the first three terms in (18), we will obtain a very tight approximation of the exact SEP written as

$$P_{pro}(\gamma) = g_1 Q(\sqrt{2A\gamma}) + \frac{4}{M}Q(2\sqrt{A\gamma}) - g_2 Q^2(\sqrt{2A\gamma}). \tag{21}$$

Note that, when $M = 32$, (21) is also the exact SEP. On the other hand, the first term of (21) is just the average number of nearest neighbors or the Gilbert approximation. At SNR = 0 (i.e., $\gamma = 0$), especially, Gilbert approximation is $2 - 3/\sqrt{2M}$, which is much larger than one, while (21) yields $P_{pro} = 1 - 1/M$, which is the exact SEP.

# 4. Symbol Error Probability in Fading Channels

The fading channels, including Rayleigh, Nakagami-m, Nakagami-q (Hoyt), and Nakagami-n (Rice) channels, are considered in this section. The probability density function (pdf) of the instantaneous received SNR $\gamma$ in these channels can be written as, respectively, [21]

$$\begin{aligned}
p_{\gamma_R}(\gamma) &= \frac{1}{\Omega}\exp(-\frac{\gamma}{\Omega}),\\
p_{\gamma_m}(\gamma) &= \frac{m^m \gamma^{m-1}}{\Omega^m \Gamma(m)}\exp(-\frac{m}{\Omega}\gamma), \quad m \ge \frac{1}{2},\\
p_{\gamma_q}(\gamma) &= \frac{1+q^2}{2q\Omega}\exp(-\frac{(1+q^2)^2}{4q^2\Omega}\gamma)I_0(\frac{1-q^4}{4q^2\Omega}\gamma),\\
&\qquad 0 \le q \le 1,\\
p_{\gamma_n}(\gamma) &= \frac{1+K}{\Omega}\exp(-K - \frac{1+K}{\Omega}\gamma)I_0(2\sqrt{\frac{1+K}{\Omega}K\gamma}),\\
&\qquad K \ge 0,
\end{aligned} \tag{22}$$

where $\gamma = r^2 E_s / N_0$ is the instantaneous SNR of the received symbol, $r$ is the instantaneous fading amplitude of the channel, $E_s$ is the energy of each transmitted symbol, $N_0$ is the one-sided power spectral density of the zero-mean AWGN, $\Omega = E\{\gamma\}$ is the average received SNR per symbol, and $E\{\cdot\}$ denotes the expectation operator. And $I_0(\cdot)$ is the modified Bessel function of the first kind and zeroth order. The moment generating function (MGF) $M_\gamma(s) \triangleq E\{e^{-s\gamma}\}$ corresponding to (22) is, respectively, given by [21]

$$\begin{aligned} M_{\gamma_R}(s) &= (1+\Omega s)^{-1}, \\ M_{\gamma_m}(s) &= \left(1+\frac{\Omega}{m}s\right)^{-m}, \quad m \geq \frac{1}{2}, \\ M_{\gamma_q}(s) &= \left(1+2\Omega s+\left(\frac{2q\Omega}{1+q^2}\right)^2 s^2\right)^{-1/2}, \quad 0 \leq q \leq 1, \\ M_{\gamma_n}(s) &= \frac{1+K}{1+K+\Omega s}\exp\left(-\frac{K\Omega s}{1+K+\Omega s}\right), \quad K \geq 0. \end{aligned} \tag{23}$$

Averaging the SEP expression (18) over the fading distribution of the received SNR, $\gamma$, induces the average SEP of arbitrary $M$-ary cross-QAM over fading channel as given by

$$\begin{aligned} P_{exa} &= \int_0^\infty P_{exa}(\gamma)p_\gamma(\gamma)d\gamma \\ &= g_1 I\left(A,\frac{\pi}{2}\right)+\frac{4}{M}I\left(2A,\frac{\pi}{2}\right)-g_2 I\left(A,\frac{\pi}{4}\right) \\ &\quad -\frac{8}{M}\sum_{k=1}^{\nu-1} I(A,\alpha_k^+)-\frac{4}{M}\sum_{k=1}^{\nu-1} I(2k^2A,\beta_k^+) \\ &\quad +\frac{4}{M}\sum_{k=2}^{\nu} I(2k^2A,\beta_k^-), \end{aligned} \tag{24}$$

where $p_\gamma(\gamma)$ denotes the PDF of $\gamma$ and

$$I(B,\phi) = 2\int_0^\infty Q_a(\sqrt{2B\gamma},\phi)p_\gamma(\gamma)d\gamma. \tag{25}$$

Note that the relationships $Q(x) = 2Q_a(x,\pi/2)$ and $Q^2(x) = 2Q_a(x,\pi/4)$ for $x \geq 0$, mentioned above, are applied in the expression (24). Particularly, the approximate average SEP is only considering the first three terms in (24).

According to (4), it is possible to re-express the integral (25) in terms of the MGF of $\gamma$ as given by

$$\begin{aligned} I(B,\phi) &= \int_0^{\infty} \frac{1}{\pi} \int_0^{\phi} \exp\left(-\frac{B\gamma}{\sin^2\theta}\right) d\theta\, p_\gamma(\gamma)\, d\gamma \\ &= \frac{1}{\pi} \int_0^{\phi} \int_0^{\infty} \exp\left(-\frac{B\gamma}{\sin^2\theta}\right) p_\gamma(\gamma)\, d\gamma\, d\theta \\ &= \frac{1}{\pi} \int_0^{\phi} M_\gamma\left(\frac{B}{\sin^2\theta}\right) d\theta . \end{aligned} \tag{26}$$

Substituting (23) into (26), the corresponding expressions of $I(B,\phi)$ in (24) for Rayleigh, Nakagami-m, Nakagami-q (Hoyt), and Nakagami-n (Rice) channels are given as

$$I_{\gamma_R}(B,\phi) = \frac{1}{\pi} \int_0^{\phi} \frac{\sin^2\theta}{\sin^2\theta + B\Omega} d\theta, \tag{27}$$

$$I_{\gamma_m}(B,\phi) = \frac{1}{\pi} \int_0^{\phi} \left(1 + \frac{\Omega}{m} \frac{B}{\sin^2\theta}\right)^{-m} d\theta, \tag{28}$$

$$I_{\gamma_q}(B,\phi) = \frac{1}{\pi} \int_0^{\phi} \left(1 + \frac{2\Omega B}{\sin^2\theta} + \left(\frac{2q\Omega}{1+q^2}\right)^2 \frac{B^2}{\sin^4\theta}\right)^{-1/2} d\theta, \tag{29}$$

$$I_{\gamma_n}(B,\phi) = \frac{1}{\pi} \int_0^{\phi} \frac{\kappa}{\kappa + \Omega B \csc^2\theta} \exp\left(-\frac{K\Omega B}{\kappa \sin^2\theta + \Omega B}\right) d\theta, \tag{30}$$

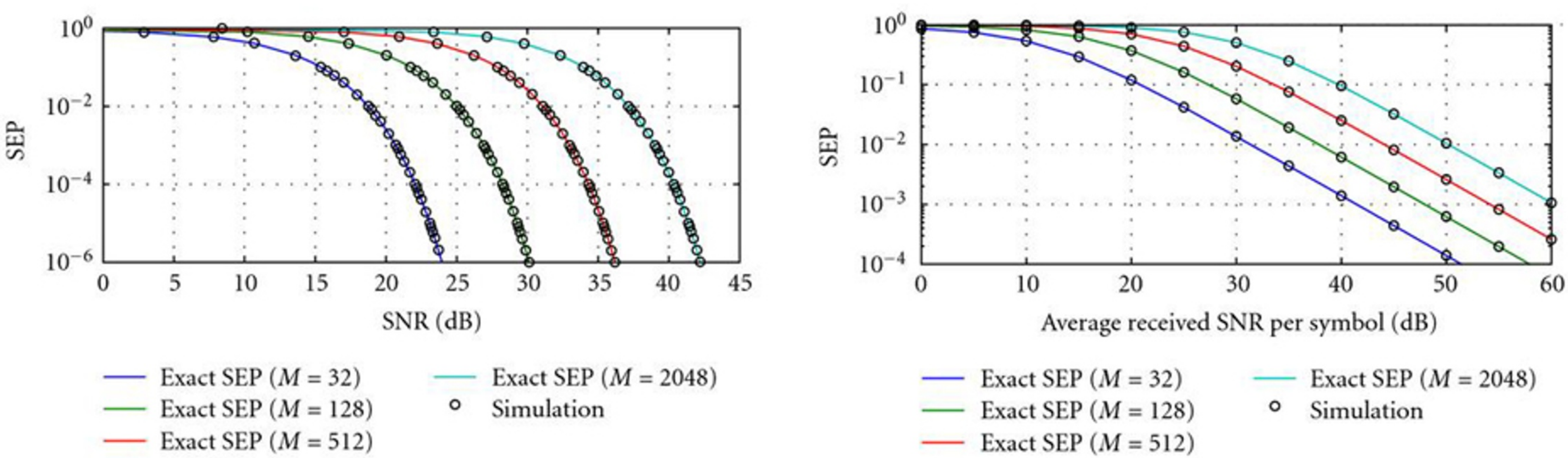
where $\kappa = 1 + K$ for the sake of notational convenience.

Using the above four integral expressions, the average SEP (24) over fading channels can be conveniently evaluated through numerical integration since these formulae are single integrals with finite limits and an integrand composed of elementary (exponential, trigonometric, and/or power) functions. Note that (27) and (28) with integer $m$ can also be evaluated in closed form using equation (5A.24) in [21, page 155]. Furthermore, the author in [20] has given a simpler closed-form expression for the SEP of cross-QAM in Rayleigh channel.

# 5. Numerical Results and Simulations

In this section, using the above formulations, numerical results concerning the SEP performance of cross-QAM and the comparison with rectangular QAM are presented, together with computer simulations. For fading channels, the Rayleigh channel is chosen as a special case since the results of the other type channels are similar to that of Rayleigh channel. So, the corresponding figures of the other type channels are not shown here for limited space. Figure 3 shows excellent agreement between the analytical expressions and the simulation results for cross $M$-QAMs over AWGN channels and Rayleigh fading channels, respectively.

**Figure 3 Comparison of the analytical and simulation results for the SEP of cross $M$-QAMs.**
AWGN ChannelsRayleigh Channels




Shown in Figure 4 is the comparison of the exact SEP and its approximations over AWGN channels and Rayleigh fading channels, respectively. In AWGN channels, it was shown that although the approximations of Proakis and Gilbert are accurate enough at high SNR, both of them are very loose and the former is looser at low SNR. From Figure 5(a), it can be seen that at given SEP the SNR differences between the exact SEP and the approximations of Proakis and Gilbert become smaller and smaller as SEP decreases. However, it can be observed from Figure 4(b) that the difference between the exact SEP and the approximations of Proakis and Gilbert in Rayleigh fading channels is larger than that in AWGN channels at high SNR. From Figure 5 (b), especially, it can be found that their SNR differences at the same SEP value are about 1 dB. On the other hand, the proposed approximation is very tight for all the SNR values whether in AWGN channels or in Rayleigh fading channels. In fact, since their SNR differences are almost 0 for all $M$ and all given SEP, so the proposed approximation is almost the same as the exact SEP.

**Figure 4 Comparison of the exact and approximate SEP expressions for cross $M$-QAMs.** Note that the proposed approximation is almost the same as the exact SEP.AWGN ChannelsRayleigh Channels

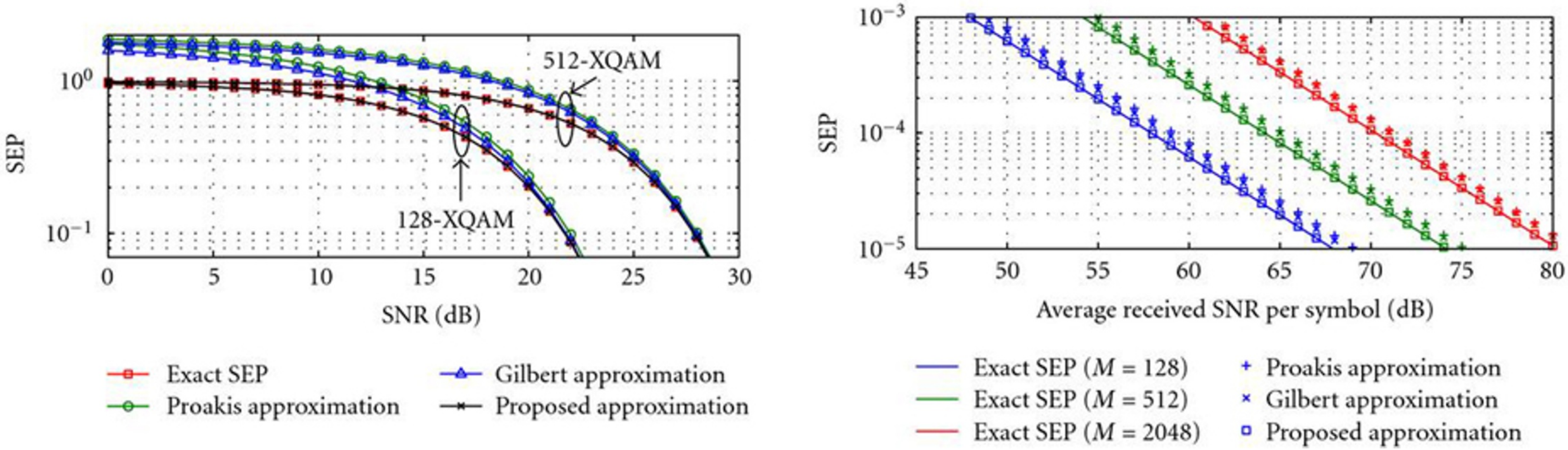



**Figure 5 The SNR differences between the approximate expressions and the exact expression of the SEP for cross $M$-QAMs.** AWGN ChannelsRayleigh Channels

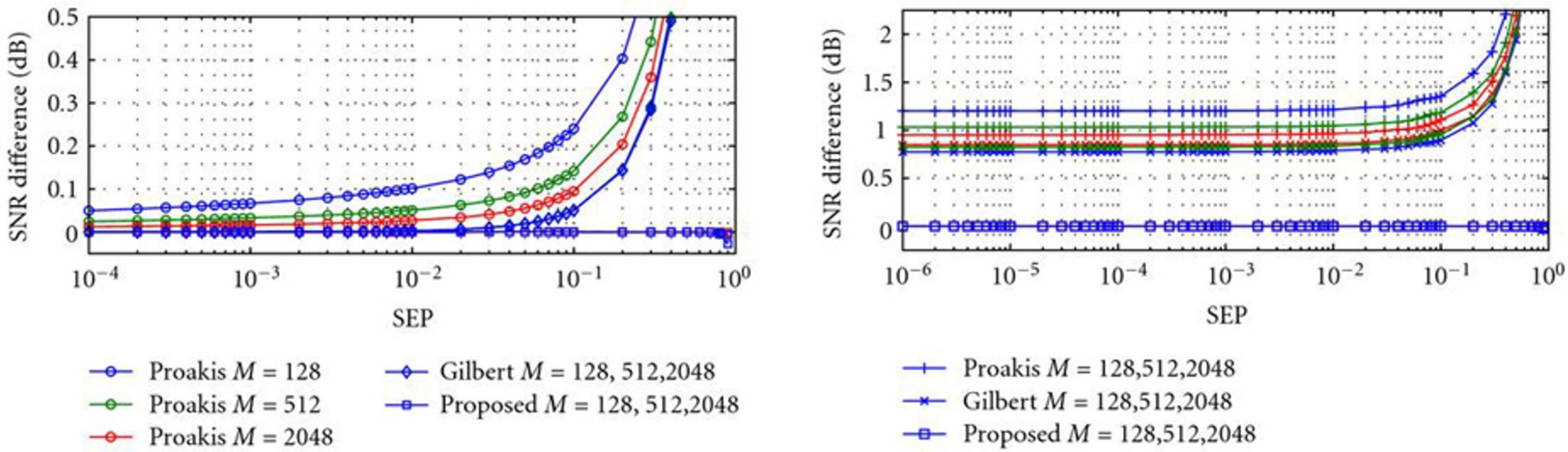



Figure 6 compares the exact SEP of rectangular QAM and cross-QAM over AWGN channels and Rayleigh fading channels, respectively. Here, the $M$-ary rectangular QAM signal constellation is assumed to be formed by drawing the inphase and quadrature components from the independent $M_I$-ary pulse amplitude modulation (PAM) and $M_Q$-ary PAM, respectively, where $M_I = 2^{m+1}$, $M_Q = 2^m$, $M = M_I \times M_Q = 2^{2m+1}$, $m \in \mathbb{N}$, and $m \geq 2$. Specifically, $8 \times 4$-ary, $16 \times 8$-ary, $32 \times 16$-ary, and $64 \times 32$-ary rectangular QAMs are considered here. In addition, the quadrature-to-inphase decision distance ratio in these constellations is set to be 1, which is the best ratio in terms of symbol error performance as shown in [25, 26]. The exact SEP of $M$-ary rectangular QAM over AWGN and fading channels are given in [27–29]. From Figure 7, it can be seen that cross-QAM exhibits at least 1.1 dB SNR gain over rectangular QAM when SEP < 0.3. Numerical evaluation under other fading scenarios with different fading parameters, which are not shown here for limited space, also reveals similar SNR gain.

**Figure 6 SEP comparison between rectangular QAM and cross-QAM.** AWGN ChannelsRayleigh Channels

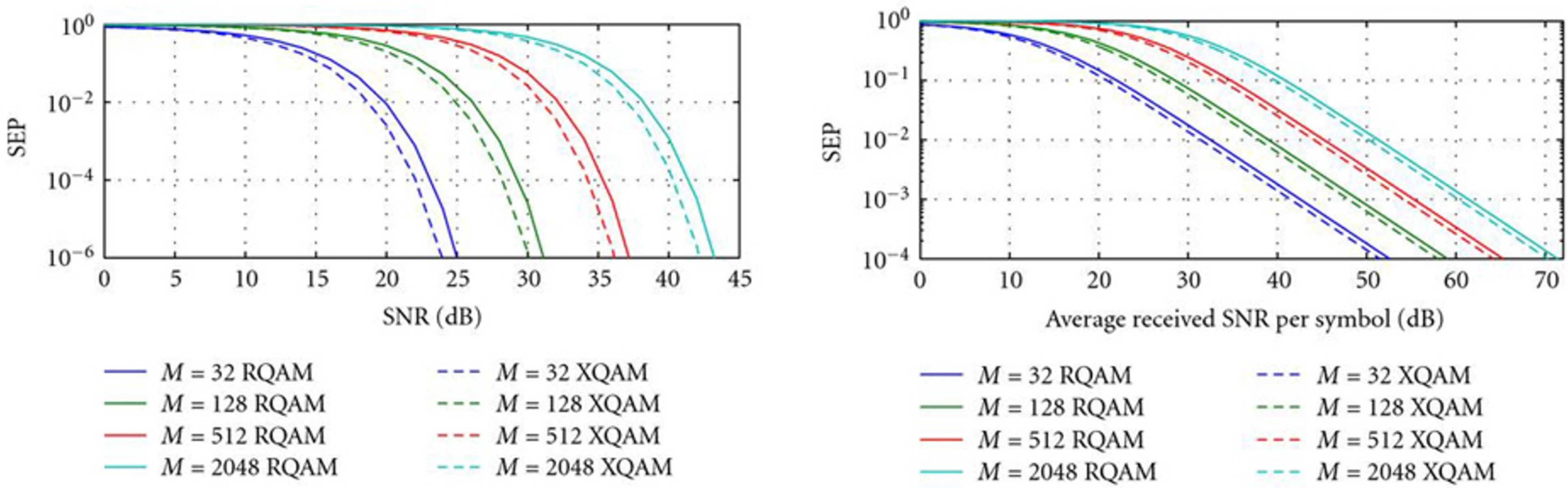
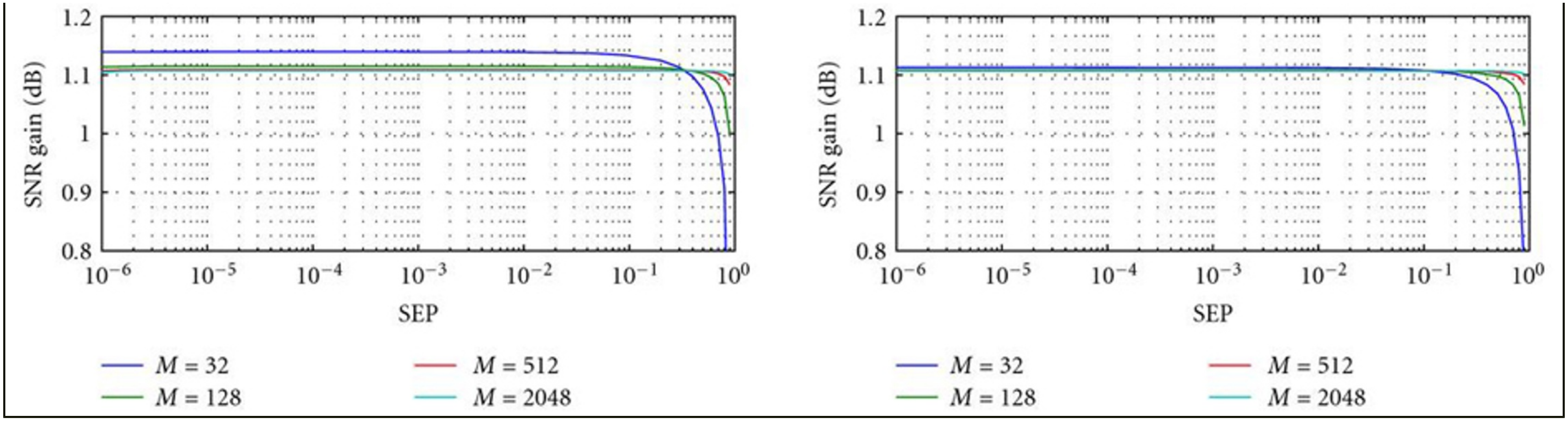



**Figure 7 SNR gain of cross-QAM over rectangular QAM.** AWGN ChannelsRayleigh Channels




# 6. Conclusions

In this paper, the exact SEP expressions of cross-QAM in AWGN channel and fading channels have been derived. The obtained closed-form SEP expressions contain a finite sum of single integrals with finite limits and an integrand composed of elementary (exponential, trigonometric, and/or power) functions, which can be easily and accurately evaluated numerically. Simple and very precise approximations, which contain only Gaussian $Q$-function for AWGN channel and contain three terms of the single integrals mentioned above for fading channels, respectively, are also given. The analytical expressions show excellent agreement with the simulation results, and the numerical evaluation with the proposed expressions reveals that cross-QAM can obtain at least 1.1 dB gain compared to rectangular QAM when SEP <0.3 in all the considered channels. The obtained exact SEP expressions will provide valuable insight into the design of wireless systems. In particular, the exact SEP performance of cross-QAM in AWGN channel will be very useful in adaptive modulation wherein the constellation size is adjusted depending on the channel quality through the SEP performance of the adopted modulations with AWGN.

# Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 61071212), Chinese Universities Scientific Fund of SCUT (nos. 2009ZM0248 and 2009ZM0271), and Nature Science Funds of Guangdong province, China (nos. 07006488 and 9351064101000003).

# Appendix

According to the definition of Gaussian $Q$-function (3), $\Phi(k,\delta;b)$ given by (16) can be represented as

$$\Phi(k,\delta;b)=\frac{1}{2\pi}\int_{-\delta}^{b}\int_{2k\delta+x}^{\infty}e^{-(x^2+z^2)/2}\,dz\,dx. \tag{A1}$$

Let $y=(z-x)/\sqrt{2}$, the above equation can be rewritten as

$$\Phi(k,\delta;b)=\frac{1}{\sqrt{2}\pi}\int_{-\delta}^{b}\int_{\sqrt{2}k\delta}^{\infty}e^{-(x^2+\sqrt{2}xy+y^2)}\,dy\,dx. \tag{A2}$$

Using the two-dimensional joint Gaussian $Q$-function [22, equation (75)],

$$\begin{aligned}Q(u,v;\rho)&\\ &=\frac{1}{2\pi\sqrt{1-\rho^2}}\\ &\quad\times\int_u^{\infty}\int_v^{\infty}\exp\left\{-\frac{(x^2+y^2-2\rho xy)}{2(1-\rho^2)}\right\}dy\,dx,\end{aligned} \tag{A3}$$

and its Simon representation [23, equation (10)] in a new form (Since $1-\rho u/v$ and $1-\rho v/u$ can take on positive or negative values, the arctangents in (A.4) are defined by $\arctan X/Y=\pi(1-\operatorname{sgn}Y)/2+\operatorname{sgn}Y\arctan(X/|Y|)$ [23].)

$$\begin{aligned}Q(u,v;\rho)\ &=Q_a\left(u,\arctan\left(\frac{\sqrt{1-\rho^2}\,u/v}{1-\rho u/v}\right)\right)\\ &\quad+Q_a\left(v,\arctan\left(\frac{\sqrt{1-\rho^2}\,v/u}{1-\rho v/u}\right)\right),\quad u\geq 0,v\geq 0,\end{aligned} \tag{A4}$$

equation (A.2) with $b=\infty$ and $b=\delta$ can be, respectively, expressed as

$$
\begin{aligned}
\Phi(k,\delta;\infty) &= 1/\sqrt{2}\pi\int_{-\delta}^{\infty}\int_{\sqrt{2}k\delta}^{\infty} e^{-(x^2+\sqrt{2}xy+y^2)}\,dy\,dx \\
&= Q(-\delta,\sqrt{2}k\delta;-\tfrac{1}{\sqrt{2}}) \\
&= Q(\sqrt{2}k\delta) - Q(\delta,\sqrt{2}k\delta;\tfrac{1}{\sqrt{2}}) \\
&= Q(\sqrt{2}k\delta) - Q_a(\delta;\alpha_k^-) - Q_a(\sqrt{2}k\delta;\pi-\beta_k^-) \\
&= Q_a(\sqrt{2}k\delta;\beta_k^-) - Q_a(\delta;\alpha_k^-), \\
\Phi(k,\delta;\delta) &= \frac{1}{\sqrt{2}\pi}\int_{-\delta}^{\delta}\int_{\sqrt{2}k\delta}^{\infty} e^{-(x^2+\sqrt{2}xy+y^2)}\,dy\,dx \\
&= \frac{1}{\sqrt{2}\pi}\int_{-\delta}^{\infty}\int_{\sqrt{2}k\delta}^{\infty} e^{-(x^2+\sqrt{2}xy+y^2)}\,dy\,dx \\
&\quad -\frac{1}{\sqrt{2}\pi}\int_{\delta}^{\infty}\int_{\sqrt{2}k\delta}^{\infty} e^{-(x^2+\sqrt{2}xy+y^2)}\,dy\,dx \\
&= \Phi(k,\delta;\infty) - Q(\delta,\sqrt{2}k\delta;-\tfrac{1}{\sqrt{2}}) \\
&= \Phi(k,\delta;\infty) - Q_a(\sqrt{2}k\delta;\beta_k^+) - Q_a(\delta;\alpha_k^+),
\end{aligned}
\tag{A5}
$$

where

$$\alpha_k^- = \arctan\left(\frac{1}{2k-1}\right), \quad k = 1,\ldots,\nu. \tag{A6}$$

Note that the property [21, equation (6.42)]

$$Q(-x,y;\rho) = Q(y) - Q(x,y;-\rho), \quad x \ge 0, y \ge 0 \tag{A7}$$

and the relationship (due to the symmetry of the sine function around $\pi/2$)

$$
\begin{aligned}
Q_a(x;\pi-\alpha) &= Q_a(x;\pi) - Q_a(x;\alpha) \\
&= Q(x) - Q_a(x;\alpha), \quad 0 \le \alpha \le \pi
\end{aligned}
\tag{A8}
$$

are used in the above derivation.With the above two equations, the exact average SEP for the $M$-ary cross-QAM constellation can be rewritten as

$$
\begin{aligned}
P_{exa}(\delta) &= \frac{4}{M}(M-6v)Q(\delta) - \frac{4}{M}(M-12v+2)Q^2(\delta) \\
&+\frac{8}{M}\Phi(v,\delta;\infty) + \frac{8}{M}\sum_{k=1}^{v-1}\Phi(k,\delta;\delta) \\
&= \frac{4}{M}(M-6v)Q(\delta) - \frac{4}{M}(M-12v+2)Q^2(\delta) \\
&+\frac{8}{M}\sum_{k=1}^{v}\left(Q_a(\sqrt{2}k\delta;\beta_k^-) - Q_a(\delta;\alpha_k^-)\right) \\
&-\frac{8}{M}\sum_{k=1}^{v-1}\left(Q_a(\sqrt{2}k\delta;\beta_k^+) + Q_a(\delta;\alpha_k^+)\right).
\end{aligned} \tag{A9}
$$

Considering that $\alpha_k^+ = \alpha_{k+1}^-$, $\alpha_1^- = \pi/4$, $Q_a(x;\pi/4) = (1/2)Q^2(x)$, one can simplifies (A.9) to yield (18) after some manipulations.

## References


1. Smith JG: **Odd-bit quadrature amplitude-shift keying.** *IEEE Transactions on Communications* 1975,**23**(3):385-389. 10.1109/TCOM.1975.1092806

2. Panigrahi S, Le-Ngoc T: **Fine-granularity loading schemes using adaptive Reed-Solomon coding for discrete multitone modulation systems.** *Proceedings of the IEEE International Conference on Communications (ICC '05), May 2005, Seoul, Korea* **2:** 1352-1356.

3. Ahrens A, Lange C: **Bit and power loading for wireline multicarrier transmission systems.** *Transactions on Advanced Research* 2006,**2**(1):3-9.

4. Zwingelstein-Colin M, Gazalet M, Gharbi M: **Non-iterative bit-loading algorithm for ADSL-type DMT applications.** *IEE Proceedings* 2003,**150**(6):414-418. 10.1049/ip-com:20030859

5. Sternad M, Falahati S: **Maximizing througput with adaptive M-QAM based on imperfect channel predictions.** *Proceedings of the IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC '04), 2004* **3:** 2289-2293.

6. Wang W, Ottosson T, Sternad M, Ahlén A, Svensson A: **Impact of multiuser diversity and channel variability on adaptive OFDM.** *Proceedings of the IEEE Vehicular Technology Conference (VTC '03), 2003* **58**(1):547-551.

7. Demjanenko V, Marzec P, Torres A: **Reasons to use non squared QAM constellations with independent I&Q in PAN systems.** submitted to IEEE P802.15-15-03, 0311-00-003a, August 2003, http://www.vocal.com/white_paper/15030311.pdf

8. Asymmetric digital subscriber line (ADSL) transceivers ITU-T Std. G.992.1, June 1999

9. Very high speed digital subscriber line transceivers ITU-T Std. G.993.1, June 2004

10. Digital Video Broadcasting (DVB) : **Framing Structure, Channel Coding and Modulation for Cable Systems.** ETSI Std. EN 300 429, April 1998, http://www.etsi.org/deliver/etsi_en/300400_300499/300429/01.02.01_60/en_300429v010201p.pdf ETSI Std. EN 300 429, April 1998, http://www.etsi.org/deliver/etsi_en/300400_300499/300429/01.02.01_60/en_300429v010201p.pdf

11. Colonnese S, Panci G, Rinauro S, Scarano G: **High SNR performance analysis of a blind frequency offset estimator for cross QAM communication.** *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP '08), March 2008, Las Vegas, Nev, USA* 2825-2828.

12. Cartwright KV, Kaminsky EJ: **Blind phase recovery in cross QAM communication systems with the Reduced-Constellation Eighth-Order Estimator (RCEOE).** *Proceedings of the IEEE Global Telecommunications Conference (GLOBECOM '05), November 2005, St. Louis, Mo, USA* **1:** 388-392.

13. Abrar S, Qureshi IM: **Blind equalization of cross-QAM signals.** *IEEE Signal Processing Letters* 2006,**13**(12):745-748.

14. Cartwright KV: **Blind phase recovery in cross QAM communication systems with eighth-order statistics.** *IEEE Signal Processing Letters* 2001,**8**(12):304-306. 10.1109/97.975873

15. Proakis JG: *Digital Communications*. 4th edition. McGraw Hill, Chicago, Ill, USA; 2001.

16. Vitthaladevuni PK, Alouini M-S, Kieffer JC: **Exact BER computation for cross QAM constellations.** *IEEE Transactions on Wireless Communications* 2005,**4**(6):3039-3050.

17. Beaulieu NC, Chen Y: **Closed-form expressions for the exact symbol error probability of 32-cross-QAM in AWGN and in slow Nakagami fading.** *IEEE Communications Letters* 2007,**11**(4):310-312.

18. Li J, Zhang XD, Beaulieu NC: **Precise calculation of the SEP of 128- and 512-Cross-QAM in AWGN.** *IEEE Communications Letters* 2008,**12**(1):1-3.

19. Liu HY, Yen RY: **Error probability for orthogonal space-time block code diversity system using rectangular QAM transmission over Rayleigh fading channels.** *IEEE Transactions on Signal Processing* 2006,**54**(4):1230-1241.

20. Yu H, Wei G: **Symbol error probability of cross QAM in rayleigh fading channels.** *IEEE Communications Letters* 2010,**14**(5):375-377.

21. Simon M, Alouini M-S: *Digital Communication over Fading Channels*. 2nd edition. John Wiley & Sons, New York, NY, USA; 2005.

22. Simon MK: **Some new twists to problems involving the gaussian probability integral.** *IEEE Transactions on Communications* 1998,**46**(2):200-210. 10.1109/26.659479

23. Simon MK: **A simpler form of the craig representation for the two-dimensional joint Gaussian Q-function.** *IEEE Communications Letters* 2002,**6**(2):49-51. 10.1109/4234.984687

24. De Abreu GTF: **Jensen-cotes upper and lower bounds on the Gaussian Q-function and related functions.** *IEEE Transactions on Communications* 2009,**57**(11):3328-3338.

25. Lei X, Fan P, Hao L: **Exact symbol error probability of general order rectangular QAM with MRC diversity reception over Nakagami-m fading channels.** *IEEE Communications Letters* 2007,**11**(12):958-960.

26. Mallik RK: **Average of product of two Gaussian Q-functions and its application to performance analysis in Nakagami fading.** *IEEE Transactions on Communications* 2008,**56**(8):1289-1299.

27. Beaulieu NC: **A useful integral for wireless communication theory and its application to rectangular signaling constellation error rates.** *IEEE Transactions on Communications* 2006,**54**(5):802-805.

28. Maaref A, Aïssa S: **Exact error probability analysis of rectangular QAM for single- and multichannel reception in Nakagami-m fading channels.** *IEEE Transactions on Communications* 2009,**57**(1):214-221.

29. Lei X, Fan P, Chen Q: **On the average of the product of two Gaussian Q-functions over Nakagami-q (Hoyt) fading and its application.** *IEEE Communications Letters* 2007,**11**(10):805-807.