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Performance Analysis of Multicarrier Systems in the Presence of Smooth Nonlinearity

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Most of the performance studies devoted to the analysis of multicarrier systems in the presence of nonlinear distortion assume that the multicarrier signal can be modeled as a random Gaussian process regardless of modulation format, guard interval duration, and number of subcarriers. On the contrary, we present an analysis based on the discrete model of the multicarrier signal. It is shown that the discrete model provides more accurate signal-to-distortion ratio and out-of-band spectrum prediction if the number of subcarriers is small. Only the effect of third-order nonlinearity is studied; however results can also be extended to the fifth-order polynomial model. Simulation results are provided which confirm analytical derivations.

Keywords and phrases: multicarrier modulation, OFDM, nonlinear distortion.

1. INTRODUCTION

Multicarrier modulation (MCM) is a technique widely used in communication systems [1]. Some of the most well-known applications are wireless local area networks (WLAN) [2], and terrestrial digital video broadcasting (DVB-T) [3] using orthogonal frequency-division multiplexing (OFDM). MCM has several advantages over single-carrier systems, those being its spectral effectiveness and reduced complexity implementation based on fast Fourier transform (FFT). A disadvantage of MCM is its high sensitivity to amplifier nonlinearity, which causes the out-of-band interference and introduces symbol error rate (SER) degradation [4, 5, 6, 7, 8].

Recently, there were many studies dealing with MCM analysis in the presence of nonlinear distortions [4, 5, 6, 7, 8]. Most of these studies are related to special models of nonlinear device (e.g., soft envelope limiter (SEL) [7], Rapp solid state power amplifier (SSPA) and Saleh traveling-wave tube amplifier (TWTA) models [4, 7], Bessel series expansion model [6, 8], etc.) and rely on assumption that the MCM signal can be modeled as a complex Gaussian process.

One problem with these methods is that the special model of nonlinear device requires expensive and time-consuming experimental measurements to identify model parameters. On the contrary, simple measures of nonlinearity directly related to a low-order polynomial model, such as third- and fifth-order intersect points, are usually available to a system designer at the early stage of specification definition or link budget analysis.

In [5], authors introduce simple analysis based on the third-order polynomial model of amplitude modulation/amplitude modulation (AM/AM) amplifier nonlinearity. The results of [5] are simple easy-to-use expressions for in-band signal-to-distortion ratio (SDR) evaluation. However, the analysis presented in [5] relies on several simplifying assumptions:

(1) Gaussian assumption on input signal,
(2) assumption that the third-order intermodulation products do not contribute to the useful signal at the input of decision device,
(3) assumption that the in-band subchannels are equally affected by nonlinearity,
(4) assumption that the nonlinear noise at the input of decision device is Gaussian.

While assumptions (1) and (4) may be reasonable, if the number of subcarriers is sufficiently large, assumptions (2) and (3) are not valid even for systems with large number of subcarriers. As a consequence, analysis presented in [5] does not show good agreement with simulation results. Recently, several papers appeared dealing with polynomial model of nonlinearity [9, 10] that are not based on simplifying assumptions (2) and (3). The analysis carried out in [10] is originally intended for code-division multiple access (CDMA) signals, but it can also be extended (with some effort) to the MCM signals. However, [9, 10] still rely on Gaussian assumption on input signal, which does not hold when the number of subcarriers is small [11].

In this paper, we propose simple discrete model for performance evaluation of MCM in low-order polynomial nonlinearity. This model eliminates the need for Gaussian assumption on input signal, and as a result can be used for SDR and out-of-band regrowth characterization when the number of MCM subcarriers is arbitrary small. We also show here that the asymptotical simplification leads to simple easy-to-use analytical formulas, suitable for systems with large number of subcarriers. Furthermore, it is expected that the results of our analysis will be helpful for characterization of non-Gaussian properties of the nonlinear distortion noise. However, this topic requires additional study and is out of scope of this paper.

It should be noted that the discrete model has already been used in past for performance characterization of the frequency-division multiple access (FDMA) satellite links [12, 13]. However, previous studies [12, 13] only examine FDMA intermodulation effects when the number of subchannels is large. Moreover, assumption made in [13] that all intermodulation products are mutually uncorrelated and contribute to a noise power is not valid for MCM systems under consideration.

Recently, there was an attempt to apply the discrete model to the analysis of nonlinearly distorted multicarrier spread spectrum systems [14]. However, the analysis presented in [14] is only intended for binary modulation formats and relies mostly on numerical computations, rather than analytical treatment.

2. SYSTEM MODEL

In the MCM transmitter, information bits are first mapped into baseband symbols $\{S_k\}$ using phase shift keying (PSK) or quadrature amplitude modulation (QAM) signaling. During active symbol interval a block of N complex baseband symbols is transformed by means of the inverse discrete Fourier transform (DFT) and a digital-to-analog conversion to the baseband MCM signal

$$z(t) = \sum_{k=0}^{N-1} g(t) S_k e^{j2\pi k \Delta f t}, \quad 0 < t < T_s, \tag{1}$$

where $g(t)$ is the signal pulse shape, N is the number of subcarriers, Δf is the separation between adjacent subcarriers, and T_s is the active symbol interval.

The third-order memoryless nonlinearity can be described by the Taylor series [15, page 13]

$$y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + a_3 x^3(t), \tag{2}$$

where $x(t)$ is the input passband signal, $y(t)$ is the output passband signal, and $\{a_n\}$ are the Taylor series coefficients. It is important to note that only the odd-order (e.g., third-order) terms produce intermodulation components that are located near central frequency of the multicarrier signal. The even-order terms are filtered out in zonal filter and do not influence system performance.

The nonlinearity of radio frequency circuits is often expressed in terms of the third-order intercept point (A_{IP3}). It can be shown [15, page 20] that A_{IP3} and parameters of model (2) are related as

$$A_{\mathrm{IP3}} = \sqrt{\frac{4a_1}{3|a_3|}}. \tag{3}$$

In this study, we only consider the third-order model of nonlinearity, since the third-order nonlinearity is usually dominated in a real system under small-signal condition [15]. However, the analysis presented here can also be extended to the fifth-order polynomial model.

3. THEORETICAL BACKGROUND

Consider the output of a nonlinear amplifier given by (2). The input signal $x(t)$ can be expressed in terms of the baseband MCM signal as

$$\begin{aligned} x(t) &= \rho(t)\cos\left[\omega_0 t + \varphi(t)\right] = \mathrm{Re}\left[z(t)e^{j\omega_0 t}\right] \\ &= \frac{1}{2}\left[z(t)e^{j\omega_0 t} + z^*(t)e^{-j\omega_0 t}\right]. \end{aligned} \tag{4}$$

In order to compute the third power of $x(t)$ one can use binomial expansion (see, e.g., [16, 17]). Combining (4) with amplifier model (2), and assuming that the components around $3\omega_0$ frequency are filtered out in zonal filter, yields the amplifier output signal [16]

$$y(t) = \mathrm{Re}\left[a_1 z(t) e^{j\omega_0 t} + \frac{3}{4} a_3 z^2(t) z^*(t) e^{j\omega_0 t}\right]. \tag{5}$$

After some manipulations and substituting (1) into (5), $y(t)$ can be expressed as (assuming for simplicity that the signal pulse shape $g(t)$ is rectangular)

$$\begin{aligned} y(t) = \mathrm{Re}\Bigg[\Bigg\{ & a_1 \sum_{k=0}^{N-1} S_k e^{j2\pi k \Delta f t} \\ & + \frac{3}{4} a_3 \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \sum_{n_3=0}^{N-1} S_{n_1} S_{n_2} S_{n_3}^* \\ & \times e^{j2\pi(n_1+n_2-n_3)\Delta f t} \Bigg\} e^{j\omega_0 t} \Bigg]. \end{aligned} \tag{6}$$

It is seen from (6) that the kth transmitted baseband symbol after transformation in a nonlinear amplifier (S'_k, $k = -N+1, -N+2, \ldots, 2N-2$) can be expressed as

$$S'_k = \begin{cases} a_1 S_k + \dfrac{3}{4} a_3 \displaystyle\sum_{n_1+n_2-n_3=k} S_{n_1} S_{n_2} S_{n_3}^* & \text{if } 0 \le k < N, \\ \dfrac{3}{4} a_3 \displaystyle\sum_{n_1+n_2-n_3=k} S_{n_1} S_{n_2} S_{n_3}^* & \text{if } -N < k < 0, \\ & \text{or } N \le k < 2N-1, \end{cases} \tag{7}$$

where n_1, n_2, n_3 can take any values in the range $0, 1, \ldots, N-1$, and $k > N-1$ or $k < 0$ corresponds to the out-of-band distortion components.

It can be shown (see the appendix for details) that the total number of intermodulation terms that correspond to the frequency $\omega_0+2\pi k\Delta f$ ($k = n_1+n_2-n_3$) can be expressed as

$$M_k^{(\text{total})} = \begin{cases} \dfrac{(N+k)(N+k+1)}{2} & \text{if } -N < k < 0, \\ \dfrac{N(N+1)}{2} + k(N-k-1) & \text{if } 0 \le k < N, \\ \dfrac{(2N-k-1)(2N-k)}{2} & \text{if } N \le k < 2N-1. \end{cases} \tag{8}$$

Now, it can be noted that the intermodulation terms can be divided into three groups defined as follows.

(1) The first group includes the terms that satisfy the following conditions:

$$(n_1+n_2-n_3 = k), \qquad ((n_1 = n_3) \text{ or } (n_2 = n_3)). \tag{9}$$

(2) The second group includes the terms that satisfy the following conditions:

$$(n_1+n_2-n_3 = k),\ (n_1 \neq n_3),\ (n_2 \neq n_3),\ (n_1 = n_2). \tag{10}$$

(3) The terms in the third group satisfy the condition $n_1 + n_2 - n_3 = k$, but do not belong to the first or second group.

It is easy to see that the terms in the first group produce scaled replica of the symbol S_k, since $S_{n_1}S_{n_2}S_{n_3}^* = S_k|S_{n_3}|^2$ or $S_{n_1}S_{n_2}S_{n_3}^* = S_k|S_{n_2}|^2$. The number of terms in the first group can be expressed as (see the appendix)

$$M_k^{(1)} = \begin{cases} 2N-1 & \text{if } 0 \le k < N, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The terms in the second and the third groups are uncorrelated with the useful signal S_k and produce both in-band and out-of-band distortions. However, the intermodulation terms in the third group are coupled in coherent pairs, since every term $S_{n_1}S_{n_2}S_{n_3}^*$ in (7) corresponds to identical term $S_{n_2}S_{n_1}S_{n_3}^*$, if $n_1 \neq n_2$. On the other hand, each term in the second group is unique. It is easier to calculate the number of terms in the second group (see the appendix). It is shown that

$$M_k^{(2)} = \begin{cases} \left\lfloor \dfrac{N+k+1}{2} \right\rfloor & \text{if } -N < k < 0, \\ \dfrac{N-2}{2} + \dfrac{1-(-1)^N}{4}(-1)^k & \text{if } 0 \le k < N, \\ \left\lfloor N - \dfrac{k}{2} \right\rfloor & \text{if } N \le k \le 2N-1, \end{cases} \tag{12}$$

where $\lfloor x \rfloor$ means the largest integer not greater than x. Finally, the number of coherent pairs in the third group can be expressed as

$$M_k^{(3)} = \frac{M_k^{(\text{total})} - M_k^{(1)} - M_k^{(2)}}{2}. \tag{13}$$

Now, using discrete signal representation (7) and formulas (8), (11), (12), and (13), we can calculate the SDR for each subcarrier of the MCM signal, and evaluate the average emitted power for every out-of-band subchannel ($k < 0$ and $k > N-1$).

4. MCM SIGNAL SPECTRUM AT THE OUTPUT OF NONLINEAR AMPLIFIER

In order to calculate the power spectral density (PSD) of the MCM signal at the output of a nonlinear amplifier we first express the average signal power in kth subchannel after nonlinear amplification, that is, we will find $P_k = E\{|S'_k|^2\}$, where $E(\cdot)$ denotes expectation. Since intermodulation terms in the first, second, and third groups are mutually uncorrelated we can perform averaging for every group separately. However, terms in the first group produce scaled replica of S_k, so they affect the power of useful signal. It was mentioned above that any intermodulation term belonging to the first group can be expressed as $(3/4)a_3S_{n_1}S_{n_2}S_{n_3}^* = (3/4)a_3S_k|S_{n_3}|^2$. Taking this fact into account and performing statistical averaging, we can easily obtain the useful signal power

$$P_k^{(1)} = \left(a_1+\frac{3}{4}a_3E\{|S|^2\}M_k^{(1)}\right)^2 E\{|S|^2\}, \quad k=0,1,\dots,N-1. \tag{14}$$

A second group term ($n_1 = n_2$) can be represented as $(3/4)a_3S_{n_1}S_{n_2}S_{n_3}^* = (3/4)a_3(S_{n_1})^2S_{n_3}^*$. Using this representation, the average power introduced by the terms in the second group can be expressed as

$$P_k^{(2)} = \left(\frac{3}{4}\right)^2 a_3^2E\{|S|^2\}E\{|S|^4\}M_k^{(2)}, \quad k = -N+1, -N+2, \dots, 2N-2. \tag{15}$$

As it was mentioned previously, the terms in the third group ($n_1 \neq n_2$) are coupled in coherent pairs, thus the average power introduced by the terms in the third group can be expressed as

$$P_k^{(3)} = \left(\frac{3}{2}\right)^2 a_3^2\left(E\{|S|^2\}\right)^3 M_k^{(3)}, \quad k = -N+1, -N+2, \dots, 2N-2. \tag{16}$$

Then, it is straightforward to represent P_k as

$$P_k = P_k^{(1)}+P_k^{(2)}+P_k^{(3)}, \quad k=-N+1, -N+2, \dots, 2N-2. \tag{17}$$

It is more convenient to express the total signal power per subchannel as a function of normalized third-order intercept point, that is, $\text{IP3} = A_{\text{IP3}}/\sqrt{E_{\text{MCM}}}$, where E_{MCM} is the total MCM signal power given by $E_{\text{MCM}} = N \times E\{|S|^2\}$. Using (3), (14), (15), (16), and (17) and performing simple manipulations, we obtain

$$P_k = \begin{cases} K_0\left(\left[1 \pm \dfrac{M_k^{(1)}}{\text{IP3}^2 N}\right]^2 + \dfrac{1}{\text{IP3}^4 N^2}\left[4M_k^{(3)} + \beta M_k^{(2)}\right]\right) & \text{if } 0 \le k < N, \\ K_0 \dfrac{1}{\text{IP3}^4 N^2}\left[4M_k^{(3)} + \beta M_k^{(2)}\right] & \text{if } -N < k < 0 \text{ or } N \le k < 2N-1, \end{cases} \tag{18}$$

where the sign ($\pm$) corresponds to a sign of coefficient a_3 (usually minus), $K_0 = a_1^2 E\{|S|^2\}$, and β is constellation dependent coefficient defined as

$$\beta = \frac{E\{|S|^4\}}{\left(E\{|S|^2\}\right)^2}. \tag{19}$$

For example, if the transmitted symbols S_k belong to an m-ary QAM constellation format with $m = 2^{2p}$, where $p = 0, 1, 2, \ldots$, and in-phase and quadrature components $I, Q = \{-(\sqrt{m}-1), -(\sqrt{m}-3), \ldots, +(\sqrt{m}-3), +(\sqrt{m}-1)\}$, β can be expressed as

$$\beta = \frac{7m - 13}{5m - 5}. \tag{20}$$

It can be seen from (18) that the average power per subchannel after nonlinear amplification (P_k) depends on the number of subcarriers (N) and the constellation type (β). However, in case of a large number of subcarriers ($N \to \infty$), it can easily be shown that P_k only depends on IP3 and ratio between k and N:

$$\lim_{N\to\infty} P_k = \begin{cases} K_0 \dfrac{2}{\text{IP3}^4}\left[1 + 2\left(\dfrac{k}{N}\right) + \left(\dfrac{k}{N}\right)^2\right] & \text{if } -N < k < 0, \\ K_0\left(\left[1 \pm \dfrac{2}{\text{IP3}^2}\right]^2 + \dfrac{1}{\text{IP3}^4}\left[1 + 2\left(\dfrac{k}{N}\right) - 2\left(\dfrac{k}{N}\right)^2\right]\right) & \text{if } 0 \le k < N, \\ K_0 \dfrac{4}{\text{IP3}^4}\left[2 - 2\left(\dfrac{k}{N}\right) + \left(\dfrac{k}{N}\right)^2\right] & \text{if } N \le k < 2N-1. \end{cases} \tag{21}$$

The PSD of the MCM signal at the output of a nonlinear amplifier depends on the power per subchannel P_k and spectral characteristics of the pulse shape $g(t)$. If the MCM system employs symbol randomization and interleaving, the sequence of symbols S'_k (for given subchannel k) is a sequence of independent identically distributed zero-mean random numbers. Thus, the PSD corresponding to the kth subchannel is given by [18, page 205]

$$\Phi_k(f) = \frac{P_k}{2T_{\text{MCM}}}\left|G(f + 2\pi k\Delta f)\right|^2, \tag{22}$$

where $G(f)$ is the Fourier transform of $g(t)$, and T_{MCM} is the total duration of the multicarrier symbol (including guard interval). Using (22), the PSD of the nonlinearly amplified MCM signal can easily be expressed as

$$\Phi(f) = \sum_{k=-N+1}^{2N-2} \frac{P_k}{2T_{\text{MCM}}}\left|G(f + 2\pi k\Delta f)\right|^2. \tag{23}$$

It is noteworthy that the PSD representation (23) takes into account number of subcarriers, guard interval duration and modulation format, and it is also accurate for systems with small number of subcarriers.

5. SIGNAL-TO-DISTORTION RATIO ANALYSIS

It can be seen from the previous section that the nonlinearity causes the out-of-band spectral regrowth and also introduces the in-band distortions. The SDR per MCM subcarrier can be calculated as a ratio between the useful signal power after nonlinear amplification given by (14) and the average power of the terms in the second and the third groups given by (15) and (16). Using this definition, it is straightforward to obtain

$$\text{SDR}_k = \frac{\left[1 \pm (\text{IP3}^2 N)^{-1} M_k^{(1)}\right]^2}{(\text{IP3}^4 N^2)^{-1}\left(4M_k^{(3)} + \beta M_k^{(2)}\right)}, \quad k = 0, 1, \ldots, N-1. \tag{24}$$

However, it should be noted that (24) represents the exact SDR only if the MCM system employs constant amplitude modulation, for example, m-ary PSK or 4-QAM. In case of m-ary QAM modulation ($m > 4$) the amplification introduced by the first group terms depends on the transmitted data sequence, and is generally random. Since a conventional MCM receiver does not take into account this effect, it is viewed by the receiver as an additional noise. Fortunately, this effect becomes insignificant when the number of subcarriers increases. More detailed analysis of this effect is provided in the next section.

When the number of subcarriers is large one can use approximate SDR expressed as (see also previous section)

$$\lim_{N\to\infty} \text{SDR}_k = \frac{\text{IP3}^4 \pm 4\text{IP3}^2 + 4}{1 + 2(k/N) - 2(k/N)^2}, \quad k = 0, 1, \ldots, N-1, \tag{25}$$

where the sign ($\pm$) corresponds to a sign of a_3.

It is interesting to note that in accordance with (25) the difference between SDR value for the central subcarrier ($k = N/2$) and the edge subcarriers ($k = 0$ or $k = N-1$) asymptotically approaches $10\log_{10}(2/3) \approx -1.76$ dB.

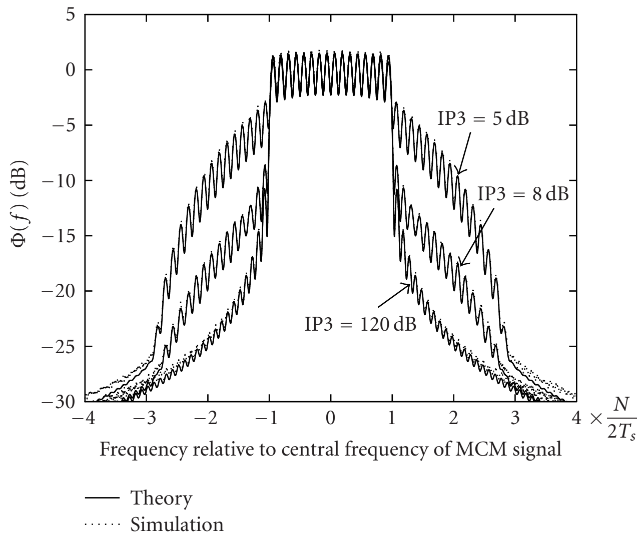

FIGURE 1: PSD of nonlinearly amplified MCM signal (4-QAM, $N = 16$, $T_{GI} = T_S/4$, $a_3 < 0$).

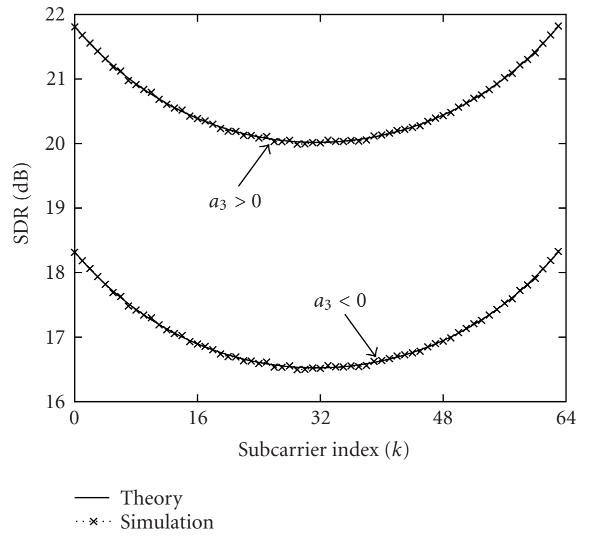

FIGURE 2: SDR versus subcarrier index for two types of nonlinearity (4-QAM, $N = 64$, IP3 = 10 dB, no background noise).

It should be mentioned that though exact SDR can be calculated for some cases (e.g., constant amplitude modulation: 4-QAM, m-ary PSK, etc.), accurate calculation of the symbol or bit error probability is more difficult task. This is due to non-Gaussian properties of the nonlinear distortion noise. Although the discrete model considered in this paper can be used to characterize non-Gaussian properties of the nonlinear noise, it is still very difficult to obtain simple analytical results for symbol or bit error rate performance. On the other hand, in the MCM systems with sufficiently large number of subcarriers, distribution of the nonlinear noise approaches Gaussian distribution (as a result of the central limit theorem). Thus, for a large N, the approximate SER can easily be calculated and averaged for all k values using well-known theoretical results for QAM or PSK performance in Gaussian noise [18, page 278].

6. NUMERICAL AND SIMULATION RESULTS

In this section, we examine effects of nonlinear distortion on the MCM performance by means of analytical calculation and by computer (Monte-Carlo) simulation. For all results below, number of simulated MCM symbols was selected so that the total number of simulated symbols (i.e., number of MCM symbols multiplied by number of subcarriers) would be 10^8.

As a first example, we investigate the PSD of the 4-QAM multicarrier system with $N = 16$, guard interval $T_{GI} = T_S/4$, and rectangular pulse shape. The numerical and simulation results for this case are presented in Figure 1. As one can see the theoretical and simulation curves are in very good agreement. It should also be noted that the proposed model can accurately predict the fine structure of the out-of-band PSD even in case of small number of subcarriers.

As a second example, we investigate the SDR and SER in the presence of the third-order distortions. In Figure 2 the SDR value versus subcarrier index for the 4-QAM MCM system with $N = 64$ and $20\log_{10}(\text{IP3}) = 10$ dB is presented. It should be noted that the two curves in Figure 2 correspond to the same value of the third-order intercept point (10 dB), the only difference is the sign of coefficient a_3. The figure emphasizes that the sign of a_3 is important for SDR evaluation, although this is sometimes ignored in the literature.

In Figure 3, the average SER (obtained analytically and by simulation) is plotted versus $20\log_{10}(\text{IP3})$ for 64-QAM MCM system with $N = 1024, 256$, and 64. As one can see proposed formulas provide very good prediction of SDR and reasonable approximation of SER when the number of subcarriers is sufficiently large. On the other hand, when $N < 256$, distribution of the nonlinear distortion noise becomes more impulsive (i.e., tails fall off more slowly then predicted by Gaussian approximation), and the actual SER becomes higher than that predicted by theory.

While Figure 3 shows results for noise-free case, it is often desirable to take account of the additive background noise. Extension of the results to this case is straightforward, since the nonlinear noise and the additive background noise are independent (see, e.g., [6, 7]). Figure 4 illustrates theoretical and simulation results, if background noise is taken into account.

It is interesting to compare the results presented in this paper with those obtained in [5]. Final result of [5] can be expressed as

$$\text{SDR} = \left(\frac{3}{2}\right)^2 \frac{\text{IP3}^4}{10}, \tag{26}$$

where we rearranged the expression in accordance with our mathematical notation.

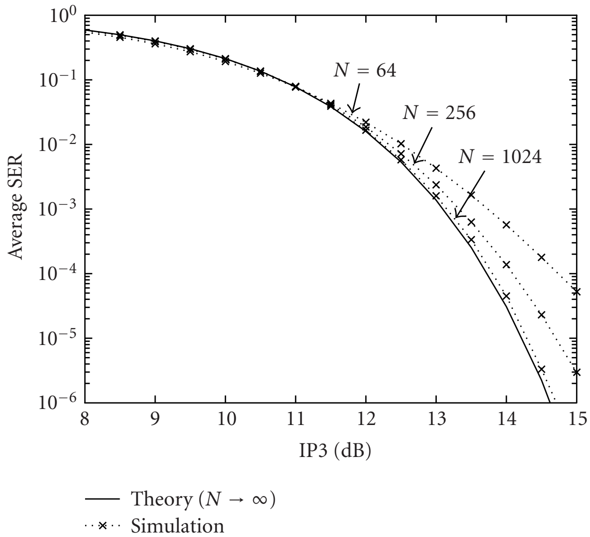

Figure 3: Average SER versus IP3: theoretical and simulation results for the finite number of subcarriers (64-QAM, $a_3 < 0$, no background noise).

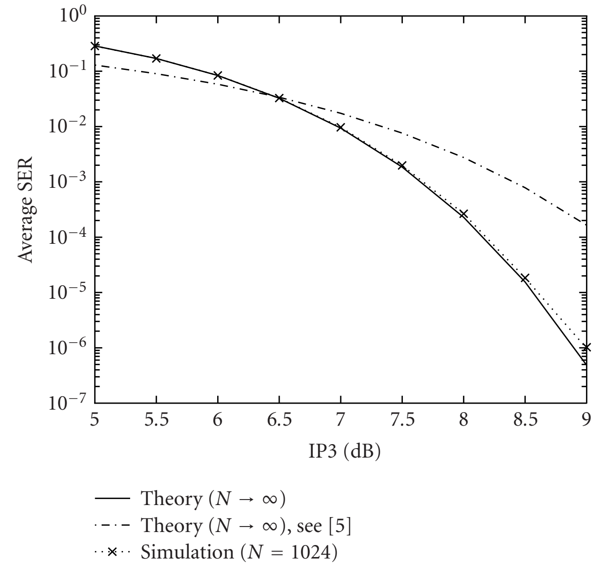

Figure 5: Average SER versus IP3: comparison with [5] (4-QAM, $a_3 < 0$, no background noise).

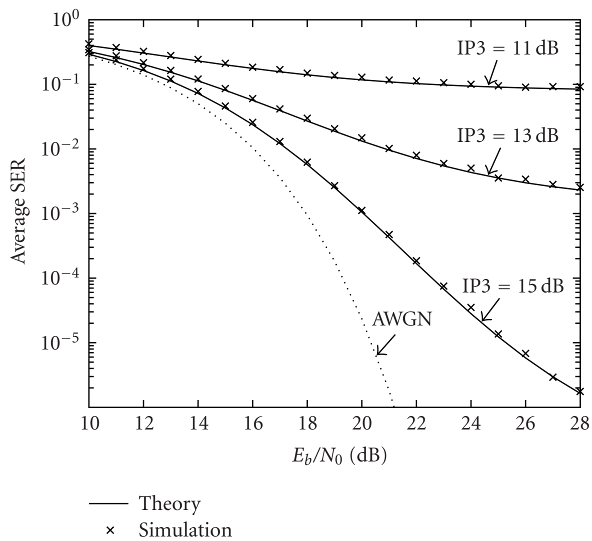

Figure 4: Average SER versus E_b/N_0 in the presence of third-order nonlinearity (64-QAM system with $N = 1024$, $a_3 < 0$).

Figure 5 illustrates comparison of simulation results (4-QAM MCM system with $N = 1024$) with the theoretical prediction obtained using (25) and (26). It is clearly seen from the illustrated example that the analysis presented in this paper provides more accurate prediction.

It is more difficult to perform comparison with other studies reported in [6, 7, 9, 10], since results of these studies are not expressed in simple form similar to (24), (25), and (26). However it is reasonable to assume that the analysis based on Gaussian assumption on input signal, and the analysis based on the discrete model provides the same results when $N \to \infty$. Thus, we can simply compare prediction obtained using (24) and asymptotical formula (25). This comparison is illustrated in Figures 6 and 7. Here, we plot SDR value for the edge subcarrier versus number of MCM subcarriers. As one can see, in case of constant amplitude modulation (4-QAM, Figure 6) theoretical prediction obtained by (24) is in perfect agreement with simulation results. In case of nonconstant amplitude modulation (16-QAM, Figure 7), however, predicted results are slightly optimistic. Thus, for nonconstant amplitude modulation (m-QAM, $m > 4$), expression (24) can be regarded as an upper bound on SDR. On the contrary, prediction obtained with asymptotical formula (25) is too pessimistic in case of small number of subcarriers. As is seen from Figure 6, SDR predicted by (25) significantly differs from the actual SDR when $N < 64$ (e.g., 5.5 dB, 2.4 dB, 1.1 dB, and 0.3 dB for $N = 4, 8, 16$, and 64, respectively).

Additional remark on the results illustrated in Figures 6 and 7 is as follows. Usually, performance of nonlinearly distorted MCM systems is only expressed in terms of SER or BER. However, in case of small number of subcarriers there are two mechanisms at work. First, SDR prediction obtained under assumption $N \to \infty$ (or Gaussian assumption on input signal) is too pessimistic when the number of subcarriers is small (see Figures 6 and 7). On the other hand, nonlinear noise at the input of decision device has strongly non-Gaussian (impulsive) behavior. Thus, BER or SER prediction using Gaussian approximation of nonlinear noise usually provides too optimistic results (see, e.g., Figure 3). These two sources of inaccuracy in some cases can simply compensate for each other. Thus, by looking at BER/SER simulation results, it may seem that the Gaussian assumption on input signal and the Gaussian approximation of nonlinear noise work well even with small number of subcarriers (see [7, 19]). However, in general it is not correct, and actual behavior of nonlinearly distorted MCM signals with small number of subcarriers is more complicated.

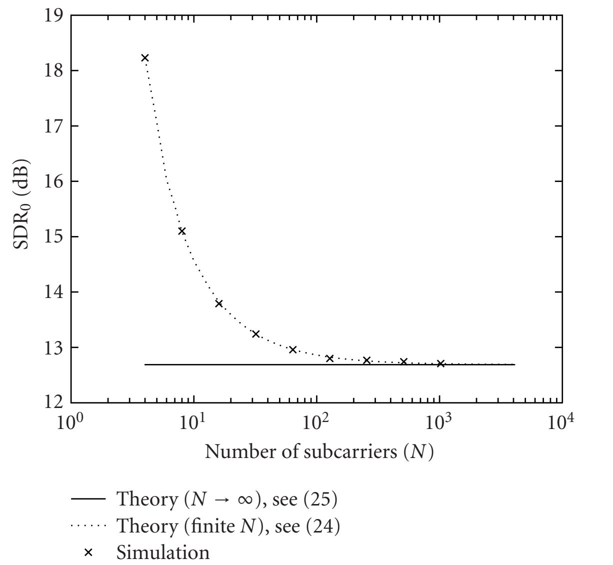

FIGURE 6: SDR for edge subcarrier of MCM signal versus number of subcarriers (4-QAM, IP3 = 8 dB, $a_3 < 0$, no background noise).

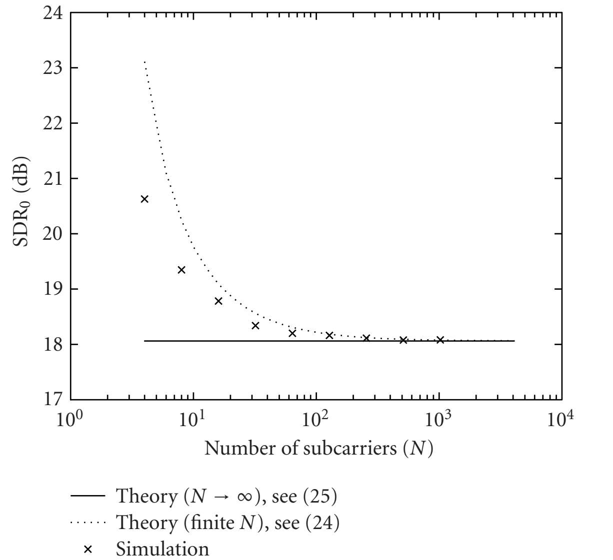

FIGURE 7: SDR for edge subcarrier of MCM signal versus number of subcarriers (16-QAM, IP3 = 10 dB, $a_3 < 0$, no background noise).

7. CONCLUSION

In this paper, simple and accurate analytical expressions for the SDR and out-of-band spectrum characterization of MCM signals in the presence of a third-order nonlinearity are derived. The analytical derivation is based on the discrete model of the MCM signal and takes into account number of MCM subcarriers and modulation format. In addition, simple asymptotical formulas are provided which can be used as a first estimation of a maximum allowable level of the third-order nonlinearity.

The effect of third-order nonlinearity has only been taken into account, since the third-order nonlinearity is usually dominated in a real system. However, the analysis presented here can also be extended to a fifth-order polynomial model. While the presented analysis provides very good prediction of the PSD and the SDR even when the number of subcarriers is sufficiently small, SER prediction is still problematic due to non-Gaussian character of the nonlinear distortion noise. Extension of the analysis to the fifth-order nonlinearity model and SER prediction improvement for the MCM systems with small number of subcarriers are subjects for future research.

APPENDIX

Derivation of (8). Consider the total number of combinations ($M_k^{(\text{total})}$) for which following condition (A.1) holds:

$$n_1 + n_2 - n_3 = k, \tag{A.1}$$

where n_1, n_2, n_3 can take any values in the range $0, 1, \ldots, N-1$, and $-N < k < 2N - 1$.

It can easily be shown that the problem of finding $M_k^{(\text{total})}$ is equivalent to the problem of derivation of the discrete convolution of the sequences $\{x_i^{(1)}\}$, $\{x_i^{(2)}\}$, and $\{x_i^{(3)}\}$, which are defined as

$$x_i^{(j)} = \begin{cases} 1 & \text{if } 0 \le i \le N-1, \\ 0 & \text{else,} \end{cases} \quad \text{for } j = 1, 2, \qquad \begin{cases} 1 & \text{if } -(N-1) \le i \le 0, \\ 0 & \text{else,} \end{cases} \quad \text{for } j = 3. \tag{A.2}$$

In order to find above-mentioned convolution we first derive z-transforms of the sequences $\{x_i^{(1)}\}$, $\{x_i^{(2)}\}$, and $\{x_i^{(3)}\}$:

$$X_j(z) = \begin{cases} \sum_{i=0}^{N-1} z^{-i} & \text{if } j = 1, 2, \\ \sum_{i=0}^{N-1} z^{i} & \text{if } j = 3. \end{cases} \tag{A.3}$$

Now, z-transform of $M_k^{(\text{total})}$ can be expressed as

$$M(z) = X_1(z)X_2(z)X_3(z) = z^{-2(N-1)}\left(\sum_{i=0}^{N-1} z^i\right)^3. \tag{A.4}$$

The sum of geometrical progression in (A.4) can be expressed as

$$\begin{aligned}\left(\sum_{i=0}^{N-1} z^i\right)^3 &= (1 - z^N)^3(1 - z)^{-3} \\ &= (1 - 3z^N + 3z^{2N} - z^{3N})(1 - z)^{-3}.\end{aligned} \tag{A.5}$$

Then, the term $(1-z)^{-3}$ can be represented using MacLaurin series

$$\left(\sum_{i=0}^{N-1} z^i\right)^3 = \left(1 - 3z^N + 3z^{2N} - z^{3N}\right) \sum_{t=0}^{\infty} C_{t+2}^2 z^t, \quad \text{(A.6)}$$

where $C_n^m = n!/(n-m)!m!$ is the binomial coefficient. Substituting (A.6) into (A.4) yields

$$\begin{aligned} M(z) &= \sum_{t=0}^{\infty} C_{t+2}^2 z^{t-2N+2} - 3\sum_{t=0}^{\infty} C_{t+2}^2 z^{t-N+2} \\ &\quad + 3\sum_{t=0}^{\infty} C_{t+2}^2 z^{t+2} - \sum_{t=0}^{\infty} C_{t+2}^2 z^{t+N+2} \\ &= \sum_{t=-2N+2}^{\infty} C_{t+2N}^2 z^t - 3\sum_{t=-N+2}^{\infty} C_{t+N}^2 z^t \\ &\quad + 3\sum_{t=2}^{\infty} C_t^2 z^t - \sum_{t=N+2}^{\infty} C_{t-N}^2 z^t. \end{aligned} \quad \text{(A.7)}$$

The number of combinations $M_k^{(\text{total})}$ that satisfy condition (A.1) is determined by the sum of all z^{-k} terms in (A.7). For example, if $0 \le k < N-1$, the number of combinations ($M_k^{(\text{total})}$) can be expressed as

$$M_k^{(\text{total})} = C_{-k+2N}^2 - 3C_{-k+N}^2, \quad \text{(A.8)}$$

where $C_n^m \equiv 0$, if $n < m$. Similarly, if $-N \le k < 0$ or $N \le k < 2N-1$, $M_k^{(\text{total})}$ can be expressed as

$$M_k^{(\text{total})} = \begin{cases} C_{-k+2N}^2 - 3C_{-k+N}^2 + 3C_{-k}^2 & \text{if } -N \le k < 0, \\ C_{-k+2N}^2 & \text{if } N \le k < 2N-1. \end{cases} \quad \text{(A.9)}$$

Finally, after simple and straightforward manipulations we can easily obtain (8).

It is interesting to note that the derivation presented here for the third-order distortion products can easily be extended to a general case of mth-order polynomial model. In such a case, we should find the number of combinations that satisfy the following condition:

$$n_1 + n_2 + \cdots + n_{(m-1)/2} - n_{(m+1)/2} - \cdots - n_m = k. \quad \text{(A.10)}$$

Applying the same procedure described above, we can generalize (A.4) as

$$M(z) = z^{-((m+1)/2)(N-1)} \left(\sum_{i=0}^{N-1} z^i\right)^m. \quad \text{(A.11)}$$

Now, (A.5) can be rewritten in general form as

$$\left(\sum_{i=0}^{N-1} z^i\right)^m = \left(1 - z^N\right)^m (1-z)^{-m}. \quad \text{(A.12)}$$

Using binomial expansion for $(1-z^N)^m$ and representing $(1-z)^{-m}$ term using MacLaurin series, we obtain

$$M(z) = z^{-((m+1)/2)(N-1)} \sum_{j=0}^{m} (-1)^j C_m^j z^{jN} \times \sum_{t=0}^{\infty} C_{t+m-1}^{m-1} z^t. \quad \text{(A.13)}$$

Finally, collecting terms containing z^{-k}, one can easily express general formula for the total number of mth-order terms falling at location of kth subchannel as

$$M_{k,m}^{(\text{total})} = \sum_{j=0}^{\lfloor k/N+(N-1)(m-1)/2N \rfloor} (-1)^j C_m^j C_{m+k-jN-1+(N-1)(m-1)/2}^{m-1}, \quad \text{(A.14)}$$

where $k = -((m-1)/2)(N-1), \ldots, ((m+1)/2)(N-1)$, and $\lfloor x \rfloor$ denotes integer part of x.

Two remarks on (A.14) are as follows. First, we should mention that the asymptotical approximation of (A.14) for $N \to \infty$ was derived in [12] and termed *number of dominant mth-order terms*. However, the results presented in [12] are not suitable for MCM systems with small number of subcarriers. Second, we should also note that in [14] authors heuristically obtained $M_{k,m}^{(\text{total})}$, for $m = 3$, and $k = 0, 1, \ldots, N-1$ (i.e., in-band channels only). On the contrary, our result (A.14) holds for arbitrary N, m and both in-band and out-of-band subchannels.

Derivation of (11). In order to derive (11), we first note that the total number of combinations that satisfy the condition $n_1 = n_3$ equals N. The number of combinations that satisfy the condition $n_2 = n_3$ also equals N. However, in one case ($k = n_1 = n_2 = n_3$) both of these conditions are satisfied simultaneously. Thus, we obtain $M_k^{(1)} = 2N - 1$, where $k = 0, 1, \ldots, N-1$.

Derivation of (12). The result can easily be obtained if we rearrange the condition ($n_1 + n_2 - n_3 = k$) and ($n_1 = n_2$) as follows:

$$2n_1 = k + n_3. \quad \text{(A.15)}$$

If value of k is fixed and within the range $0, \ldots, N-1$, then the right-hand side of (A.15) can take N different values: $k, k+1, k+2, \ldots, k+N-1$. However, condition (A.15) is only satisfied if the sum in the right-hand side of (A.10) is even. Thus, it is easy to see that when value of N is even the number of combinations that satisfy (A.15) is always $N/2$ regardless of k value. However, one combination should be excluded from our consideration, because it satisfies the condition $k = n_1 = n_2 = n_3$. Thus, if $k = 0, \ldots, N-1$ and N is even then $M_k^{(2)} = N/2 - 1$. Derivation of $M_k^{(2)}$ for odd N and out-of-band subchannels is straightforward.

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