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Performance Analysis of Capacity of MIMO Systems under Multiuser Interference Based on Worst-Case Noise Behavior

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The capacity of a cellular multiuser MIMO system depends on various parameters, for example, the system structure, the transmit and receive strategies, the channel state information at the transmitter and the receiver, and the channel properties. Recently, the main focus of research was on single-user MIMO systems, their channel capacity, and their error performance with space-time coding. In general, the capacity of a cellular multiuser MIMO system is limited by additive white Gaussian noise, intracell interference from other users within the cell, and intercell interference from users outside the considered cell. We study one point-to-point link, on which interference acts. The interference models the different system scenarios and various parameters. Therefore, we consider three scenarios in which the noise is subject to different constraints. A general trace constraint is used in the first scenario. The noise covariance matrix eigenvalues are kept fixed in the second scenario, and in the third scenario the entries on the diagonal of the noise covariance matrix are kept fixed. We assume that the receiver as well as the transmitter have perfect channel state information. We solve the corresponding minimax programming problems and characterize the worst-case noise and the optimal transmit strategy. In all scenarios, the achievable capacity of the MIMO system with worst-case noise is equal to the capacity of some MIMO system in which either the channels are orthogonal or the transmit antennas are not allowed to cooperate or in which no channel state information is available at the transmitter. Furthermore, the minimax expressions fulfill a saddle point property. All theoretical results are illustrated by examples and numerical simulations.

Keywords and phrases: MIMO channel capacity, multiuser interference, worst-case noise, optimal transmission strategy.

1. INTRODUCTION

Multiple antenna systems provide high spectral efficiencies and improved performance [1, 2]. In next-generation wireless systems, multiple users equipped with multiple antennas will transmit simultaneously to a base station with multiple receive antennas. So far, the emphasis of research was on the point-to-point single-user case. The achievable rates, the tradeoff between diversity and multiplexing, and space-time coding for single-user MIMO systems were intensely studied.

The analysis of multiuser MIMO systems is very important, because usually more than one user is involved in cellular as well as ad hoc systems. Up to now, only little is known about MIMO multiuser systems. The achievable rates and the transmission strategy depend on the following.

(i) *Structure of the wireless MIMO system.* In the common cellular approach, many mobiles share one base station which controls the scheduling and transmission strategies, for example, power control in a centralized manner. In cellular systems, the intercell and intracell interference can be controlled by spectrum and time allocation. In MIMO system, an additional dimension, namely, the space, is available for allocation purposes. In ad-hoc systems, the properties of one adaptive temporarily created wireless link cannot be controlled by a central entity. One mobile could use possible surrounding mobiles to build a virtual MIMO link. In an arbitrary way, interference which is created by other mobiles building virtual MIMO links, or just transmitting, acts on one virtual MIMO link.

(ii) *Transmit strategies.* Obviously, the transmit strategies of the participating mobiles influence the achievable rate and the properties of the complete system. The transmit strategies in turn depend on the type of channel state information (CSI) at the transmitter, that is, the more CSI available about the own channel as well as about the other users and the interference, the more adaptive and smart transmission strategies can be applied. If no CSI is available at the transmitter, the use of multiuser space-time (-spreading) codes is advised.

(iii) *Receive strategies.* At the receiver, different decoding and detection strategies can be used. The range is from single-user detection algorithms which treat the other users a noise, up to linear and even nonlinear multiuser detection algorithms. Of course, the receiver architecture depends on the type of CSI, too.

(iv) *System parameters.* In general, the scenario in which the wireless system is applied is an important factor. In home or office scenarios, the system parameter heavily differ from parameters in public access, hot-spots, or high velocity scenarios. User parameters, resource parameters, and especially channel parameters have to be taken into account. The achievable performance and throughput depend on those system parameters.

The impact of interference in single-cell multiuser MIMO systems was studied in [3, 4, 5]. Joint processing leads to a set of optimal transmit covariance matrices which maximize the sum capacity. These results are only valid for perfect CSI at both sides of the link, and for successive interference cancellation (SIC) for the uplink or Costa precoding for the downlink. For independent decoding or precoding, the complicated optimization problem was studied in [6]. Under the assumption of a multiuser MMSE receiver, the minimization of the average sum MSE and the structure of the individual MSE region was analyzed in [7].

In this work, we study one point-to-point link, on which interference acts. We model the impact of the mentioned effects on the system by a special noise covariance matrix and analyze for different scenarios the structure of the capacity of the resulting MIMO channel. We do not assume a priori structure of the interference. It could be the uplink or downlink transmission, interference could be intercell or intracell interference. Receiver noise, intercell, and intracell interference restrict the achievable capacity. Therefore, it is important to study minimax expressions for the worst-case noise capacity, like

$$\min_{\mathbf{Z}\in\mathcal{Z}} \max_{\mathbf{Q}\in\mathcal{Q}} F(\mathbf{Z},\mathbf{Q}), \tag{1}$$

in which the noise covariance matrix $\mathbf{Z}$ is in some set of admissible noises $\mathcal{Z}$ and the transmit strategy, that is, the transmit covariance matrix $\mathbf{Q}$ belongs to some set of admissible transmit strategies $\mathcal{Q}$. The function F is an arbitrary objective function, for example, capacity. If F is the sum capacity, we know that

$$F(\mathbf{A},\mathbf{B}) = \log\left(\frac{\det(\mathbf{A}+\mathbf{B})}{\det\mathbf{A}}\right) \tag{2}$$

is convex in $\mathbf{A}$ (see [8, Lemma II.3]) and concave in $\mathbf{B}$ (see [8, Lemma II.4]).

In [9], expressions like (1) were studied under different admissible sets $\mathcal{Z}$ and $\mathcal{Q}$. In order to characterize the minimax points, the authors in [9] used the dual Lagrangian approach. Based on the characterization of the broadcast rate region in [10], in [11], the authors established a duality and reciprocity theory between the SIMO multiple access sum capacity point, MIMO uplink capacity, MIMO downlink capacity, and MISO broadcast sum capacity point for systems which apply SIC and Costa precoding. The duality between the SIMO multiple access channel (MAC) sum capacity point and MIMO uplink capacity corresponds to the noise constraints in Scenario III. For independent coding and decoding, the uplink-downlink duality for SIMO was shown in [12].

In [13], the author analyzes the MIMO channel capacity with unknown interfering users and no CSI at the transmitter and perfect CSI at the receiver. The optimum signaling for achieving the channel capacity is characterized by analyzing the second derivative of the mutual information and showing that in some cases it is negative and in some cases it is positive. Therefore, in these cases in which the interference is sufficiently weak or sufficiently strong, the optimum signaling is either equal power allocation across all antennas or single-antenna allocation. In [14], the mutual information of a MIMO system with multiple users and perfect CSI at both sides is considered and different signaling approaches are considered. The problem of sum capacity in MIMO MAC has been solved for fixed individual powers in [15] and for a sum power constraint in [3, 4].

We are interested in the general limits of the MIMO channel capacity under different types of noise plus interference. Therefore, we consider three scenarios in which the noise is subject to different constraints. In scenario I, the noise covariance matrix is trace constrained. This is the most generous constraint, because only the sum noise power is kept fixed. In scenario II, the eigenvalues of the noise covariance matrix are fixed. This leads to the notion of the worst-case directions which correspond to the eigenvectors of the noise covariance matrix $\mathbf{Z}$. Finally, in scenario III, the diagonal elements of the noise covariance matrix are fixed. This yields the worst-case colored noise. We show for all three scenarios that the achievable minimax capacity fulfills the saddle point property. Furthermore, the wort case noise in Scenario I and Scenario II leads to two different types of worst-case orthogonal channels. In Scenario I, the complete CSI at the transmitter is lost and therefore the cooperation, too, because CSI is necessary for successful cooperation at the transmit side. In Scenario III, the capacity of the MIMO channel with worst-case colored noise equals the capacity of a multiuser SIMO channel with white noise, that is, the transmitter cooperation[1] gets lost. The contributions of this paper are as follows.

[1]Cooperation between transmit antennas means that the spatial multiplexed data streams can be distributed in an arbitrary fashion over the antennas. For example, a V-BLAST system [16] does not need cooperation among the transmit antennas. An SVD-based approach needs cooperation.

(1) The capacity of a MIMO closed-loop system, that is, perfect CSI at the transmitter, with worst-case noise under a trace constraint (or worst-case interference) equals the capacity of a MIMO open-loop system, that is, no CSI at the transmitter, with white noise, that is, without interference. The structure of the equivalent system is a single-user MIMO system with uncorrelated noise and without CSI at the transmitter. We completely characterize the solution of the corresponding minimax expression in (1).

(2) The worst-case noise directions correspond with the left eigenvectors of the channel matrix **H**. The optimal transmit directions correspond with the right eigenvectors of the channel matrix **H**. Both are independent of each other. Therefore, the minimax problem fulfills the saddle point property. The power allocation is then the well-known waterfilling solution.

(3) The worst-case colored noise decomposes the closed-loop MIMO system into a SIMO MAC with amplified white noise. The transmitter cooperation goes loose and the noise is amplified by a factor equal to the number of receive antennas.

A minimax approach in [17] studies the maximum of the mutual information with respect to the transmit covariance matrix and the minimum with respect to the channel realization of the instantaneous capacity in a flat-fading MIMO channel. In addition to this, the worst-case capacity of a MIMO system is studied in [18]. In [19], the MIMO broadcast channel (BC) was studied and the structure of the worst-case noise of the corresponding cooperative MIMO system which minimizes the Sato upper bound was analyzed.

The paper is organized as follows: The signal model, noise scenarios, and important preliminary results are presented in the next section. In Section 3, we study the worst-case noise under a trace constraint. In Section 4, we analyze the worst-case noise direction for fixed noise covariance matrix eigenvalues and in Section 5, we investigate the worst-case colored noise problem. Interpretations, illustrations, and discussions are provided in Section 6 and we conclude the paper in Section 7.

1.1. Notation

Vectors are denoted by bold letters **a**, matrices are denoted by bold capital letters **A**. det(**A**) is the determinant of matrix **A**. rank(**A**) is the rank of the matrix **A**. tr(**A**) is the trace, that is, $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{\operatorname{rank}(\mathbf{A})} A_{ii}$. $\mathcal{E}$ is the expectation operator. All logarithms $\log(a)$ in this work are logarithms dualis. n_T is the number of transmit antennas and n_R is the number of receive antennas.

2. SYSTEM MODELS, NOISE SCENARIOS, AND PRELIMINARIES

2.1. System models

The analysis in this work can be applied to the following systems, namely, the MIMO MAC and the MIMO BC. Using the MIMO MAC, the basic assumptions are motivated. The MIMO BC can be used for motivation, too.

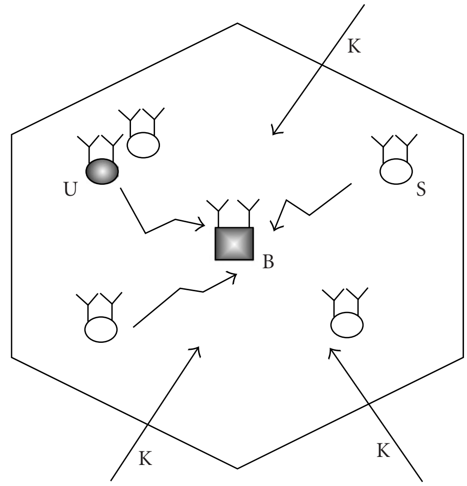

FIGURE 1: Cellular MIMO multiuser uplink.

2.1.1. MIMO MAC

We consider the typical flat-fading MIMO MAC model with n_T transmit and n_R receive antennas in cellular multiuser uplink transmission. In Figure 1, we show an uplink transmission from user (U) to the base (B). On the one hand, intercell interference comes from neighbor cells (K) and on the other hand users in the same cell create intracell interference (S).

Following the quasistatic block flat-fading cellular MIMO multiuser uplink model considered above, the received signal is given by

$$\mathbf{y} = \underbrace{\mathbf{H}\mathbf{x}}_{\text{user signal}} + \underbrace{\sum_{k\in\mathcal{S}} \mathbf{H}_k\mathbf{x}_k}_{\text{intracell interference}} + \underbrace{\sum_{k\in\mathcal{K}} \mathbf{H}_k\mathbf{x}_k}_{\text{intercell interference}} + \mathbf{n} \tag{3}$$

with white Gaussian noise $\mathbf{n} \sim \mathcal{CN}(0, \sigma^2\mathbf{I})$. We collect all noise and interference terms in one vector **z** and obtain

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}, \tag{4}$$

where **x** is the transmitted signal, **H** is the channel matrix, and **z** is the interference plus noise. The mobile has n_T transmit antennas and the base n_R receive antennas. The channel matrices and signals of the intercell and intracell interfering users can have arbitrary number of transmit dimensions.

The coherence time of the channel is large enough for coding over sufficiently many blocks. The interference during one fixed channel realization depends on the statistics of the transmitted signals of the interferers, and on their instantaneous channel realizations. Now, there are two reasons for colored noise: the first one is that the interfering users choose transmit covariance matrices which are not equal to identity. This could happen, if they adapt to their channel states. The other possibility is that the spatial structure of the channel between the interferers and the receiver creates this colored noise. Furthermore, the transmitter as well as the interferers use Gaussian codebooks in order to achieve capacity. Therefore, we assume that the interference plus noise is complex Gaussian distributed with covariance matrix $\mathbf{Z}$. Furthermore, the zero-mean complex Gaussian distribution is the worst-case noise distribution under a variance constraint. Therefore, we model the interference plus noise as zero-mean complex Gaussian distributed with covariance matrix $\mathbf{Z}$, that is, $\mathbf{z} \sim \mathcal{CN}(0, \mathbf{Z})$.

The optimum input distribution which maximizes the capacity of the channel in (4) is the zero-mean complex Gaussian distribution, too, that is, $\mathbf{x} \sim \mathcal{CN}(0, \mathbf{Q})$. The transmit covariance matrix is given by $\mathbf{Q} = \mathcal{E}(\mathbf{x}\mathbf{x}^H)$. The mutual information of the channel in (4) is [1]

$$C(\mathbf{Q}, \mathbf{Z}) = \log \frac{\det\left(\mathbf{Z} + \mathbf{H}\mathbf{Q}\mathbf{H}^H\right)}{\det(\mathbf{Z})}. \tag{5}$$

We assume that the sum transmit power is constrained to P, that is, $\operatorname{tr}(\mathbf{Q}) \le P$.

2.2. Noise scenarios

In the following, we will need the eigenvalue decomposition of the transmit, noise, and channel covariance matrices. We define $\mathbf{Z} = \mathbf{U}_Z \boldsymbol{\Lambda}_Z \mathbf{U}_Z^H$, $\mathbf{Q} = \mathbf{U}_Q \boldsymbol{\Lambda}_Q \mathbf{U}_Q^H$, $\mathbf{H}\mathbf{H}^{\mathbf{H}} = \mathbf{U}_H \boldsymbol{\Lambda}_H^1 \mathbf{U}_H^H$, and $\mathbf{H}^{\mathbf{H}}\mathbf{H} = \mathbf{V}_H \boldsymbol{\Lambda}_H^2 \mathbf{V}_H^H$. To avoid confusion with the dimension of $\boldsymbol{\Lambda}_H^1$ and $\boldsymbol{\Lambda}_H^2$, we will use the diagonal matrix $\boldsymbol{\Lambda}_H$ of dimension $\min(n_R, n_T)$ with the positive-ordered eigenvalues on the diagonal.[2] The singular value decomposition of the channel matrix $\mathbf{H}$ is given by $\mathbf{H} = \mathbf{U}_H \boldsymbol{\Lambda}_H^{1/2} \mathbf{V}_H^H$.

In all scenarios, the transmit power is constrained to P, that is, $\operatorname{tr}(\mathbf{Q}) \le P$. The spatial signature of the interfering users has a direct impact on the noise plus interference covariance matrix $\mathbf{Z}$. The eigenvectors correspond with the directions of the interfering signals and the eigenvalues correspond with the average powers which depend on the distance between interferer and receiver.

The following three scenarios are studied.

(1) *Trace constraint*. The trace of the noise covariance matrix is a constraint to $n_R \sigma_N^2$, that is,

$$\operatorname{tr}(\mathbf{Z}) \le \sigma_N^2 n_R. \tag{6}$$

In this scenario, the sum noise power which arrives at the base station is kept fixed. The noise has no additional constraints. This model corresponds to a scenario in which the intercell and intracell interference dominates. The noise has the fewest constraints in comparison to other scenarios.

(2) *Fixed eigenvalues*. The eigenvalues of the noise covariance matrix are fixed. The diagonal matrix

$$\boldsymbol{\Lambda}_Z = \operatorname{diag}\left(\lambda_1(\mathbf{Z}), \ldots, \lambda_m(\mathbf{Z})\right) \tag{7}$$

is fixed.

Here, the average power (eigenvalues of noise covariance matrix $\boldsymbol{\Lambda}_Z$) is fixed, while the dominant directions of the noise (eigenvectors of the noise covariance matrix $\mathbf{U}$) vary. This constraint leads to the worst-case noise directions.

(3) *Diagonal constraint*. The diagonal of the noise covariance matrix is constraint to be less or equal to some constant σ_N^2, that is,

$$\operatorname{diag}(\mathbf{Z}) = \left[\sigma_N^2, \ldots, \sigma_N^2\right]. \tag{8}$$

In this scenario, we fix the noise power at each receive antenna at the base station because of the equal noise power of the receivers. The color in the noise is created by the intracell and intercell interference. The free parameter is the correlation of the noise. This scenario provides the worst-case colored noise. In addition to this, this scenario is interesting from an information theoretic point of view, since this noise constraint is used to compute the upper bound on the achievable rate of a MIMO multiuser system [10].

2.3. Preliminaries

In this work, many min-max expressions occur. In order to decide whether the min-max expressions satisfy the saddle-point property

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y), \tag{9}$$

we use [20, Theorem 1]. One result in [20, Theorem 1] states that (9) is fulfilled if f is convex on X and concave on Y and if the sets X and Y are convex, too.

The capacity $C(\mathbf{Q}, \mathbf{Z})$ in (5) is convex with respect to $\mathbf{Z}$ [8, Lemma II.3] and concave with respect to $\mathbf{Q}$ [21, Theorem 1]. The set of all possible transmit covariance matrices $\mathbf{Q}$ is obviously convex. The set of admissible noise covariance matrices $\mathbf{Z}$ in Scenario I and Scenario III is convex. As a result, the min-max problem for Scenario I and Scenario III fulfill the saddle-point property. Only for Scenario II, we have to explicitly compute the min-max and max-min value in order to show that they are equal.

In order to derive upper and lower bounds for functions like $\det(\mathbf{A} + \mathbf{B})$, the following theorem is helpful [22].

Theorem 1 (Fiedler 1971). *For positive semidefinite matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *with eigenvalues* $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$ *and* $\beta_1 \ge \beta_2 \ge \cdots \ge \beta_n$, *it holds that*

$$\prod_{i=1}^{n} \left(\alpha_i + \beta_i\right) \le \det(\mathbf{A} + \mathbf{B}) \le \prod_{i=1}^{n} \left(\alpha_i + \beta_{n+1-i}\right). \tag{10}$$

Theorem 1 makes two statements. We keep the matrix $\mathbf{A}$ and its eigenvalue decomposition $\mathbf{A} = \mathbf{U}_A \boldsymbol{\Lambda}_A \mathbf{U}_A^H$ fixed and let the eigenvalue decomposition of $\mathbf{B}$ be given by $\mathbf{B} = \mathbf{U}_B \boldsymbol{\Lambda}_B \mathbf{U}_B^H$. The first statement in Theorem 1 is that the minimum and maximum of $\det(\mathbf{A}+\mathbf{B})$ is achieved by a specific choice of the eigenvectors in $\mathbf{U}_B$: $\mathbf{U}_B = \mathbf{P}(\pi)\mathbf{U}_A$ with a permutation matrix

[2] The eigenvalues in the diagonal matrix $\boldsymbol{\Lambda}_H$ can be ordered without loss of generality, since the resulting matrix depends only on the tuples of the eigenvalues and eigenvectors and not on their order.

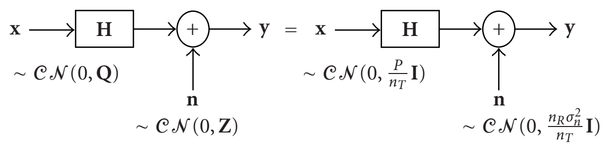

FIGURE 2: Worst-case noise with trace constraint: vector MIMO channel C_{I} and corresponding diagonalized orthogonal channels C_{I}^D.

$\mathbf{P}(\pi)$ and a permutation π. Therefore, the first statement in Theorem 1 is

$$\min_{\pi}\prod_{i=1}^{n}\left(\alpha_i+\beta_{\pi_i}\right) \leq \det(\mathbf{A}+\mathbf{B}) \leq \max_{\pi}\prod_{i=1}^{n}\left(\alpha_i+\beta_{\pi_i}\right). \quad (11)$$

The second statement in Theorem 1 specifies the minimum and maximum permutation π in (11). Using the result from [23, Proposition 3.E.1] that the product of logarithmically concave functions is Schur-concave, it follows that the minimum is attained for equally sorted α_i and β_i and the maximum is attained for oppositional sorted α_i and β_{n-i+1}.

3. WORST-CASE NOISE WITH TRACE CONSTRAINT

In this section, the worst-case noise with the trace constraint from scenario I is characterized. The optimization problem for scenario I is given by

$$C_{\text{I}} = \min_{\operatorname{tr}(\mathbf{Z})\leq\sigma_N^2 n_R}\ \max_{\operatorname{tr}(\mathbf{Q})\leq P}\log\frac{\det\left(\mathbf{Z}+\mathbf{HQH}^H\right)}{\det\mathbf{Z}}. \quad (12)$$

The problem in (12) fulfills the saddle point properties [20]. Therefore, we can switch the min-max problem into max-min, that is,

$$\min_{\operatorname{tr}(\mathbf{Z})\leq\sigma_N^2 n_R}\ \max_{\operatorname{tr}(\mathbf{Q})\leq P} C(\mathbf{Q},\mathbf{Z}) = \max_{\operatorname{tr}(\mathbf{Q})\leq P}\ \min_{\operatorname{tr}(\mathbf{Z})\leq\sigma_N^2 n_R} C(\mathbf{Q},\mathbf{Z}). \quad (13)$$

We define the following optimization problem:

$$C_{\text{I}}^D = \min_{\substack{\mathbf{\Lambda}_Z\succ 0\\ \operatorname{tr}(\mathbf{\Lambda}_Z)\leq n_R\sigma_N^2}}\ \max_{\substack{\mathbf{\Lambda}_Q\succeq 0\\ \operatorname{tr}(\mathbf{\Lambda}_Q)\leq P}}\log\frac{\det\left(\mathbf{\Lambda}_Z+\mathbf{\Lambda}_H\mathbf{\Lambda}_Q\right)}{\det\mathbf{\Lambda}_Z} \quad (14)$$

with noise eigenvalues $\mathbf{\Lambda}_Z = \operatorname{diag}[\lambda_1(\mathbf{Z}),\ldots,\lambda_{n_R}(\mathbf{Z})]$ ordered in decreasing order, that is, $\lambda_1(\mathbf{Z}) \geq \lambda_2(\mathbf{Z}) \geq \cdots \geq \lambda_{n_R}(\mathbf{Z})$. The problem in (14) fulfilled the saddle point property, too. Therefore, the minimization and the maximization in (14) can be switched.

We describe our first result. The vector MIMO channel with perfect CSI at the transmitter transforms into a MIMO channel without CSI and white Gaussian noise.

In Figure 2, the correspondence is shown between the closed-loop MIMO system (transmit covariance matrix $\mathbf{Q}$) with worst-case noise (noise covariance matrix $\mathbf{Z}$) with trace constraint and the open-loop MIMO system (transmit covariance matrix $\mathbf{I}$) with white noise. Let m denote the minimum of the number of transmit and receive antennas, that is, $m = \min(n_T, n_R)$.

The following theorem recapitulates this correspondence.

Theorem 2. *The saddle point of the minimax problem C_{I} and C_{I}^D equals and is given by*

$$\begin{aligned} C_{\text{I}} = C_{\text{I}}^D &= \sum_{k=1}^{m}\log\left(1+\frac{\lambda_i(\mathbf{H})\lambda_i^*(\mathbf{Q})}{\lambda_i^*(\mathbf{Z})}\right)\\ &= \sum_{k=1}^{m}\log\left(1+\rho\lambda_i(\mathbf{H})\right)\\ &= \log\det\left(\mathbf{I}+\rho\mathbf{HH}^H\right)\\ &= \log\det\left(\mathbf{I}+\frac{P}{\sigma_n^2 n_R}\mathbf{HH}^H\right). \end{aligned} \quad (15)$$

The capacity in (15) *equals the capacity of an open-loop MIMO system, that is, without CSI at the transmitter, without interference, that is, with white uncorrelated noise with an effective SNR $\rho = P/(n_R\sigma_N^2)$.*

Proof. The proof consists of two parts. In the first part, we show that the capacity in (12) and the capacity in (14) are equal.

Lemma 1. *The capacity C_{I} in* (12) *and the capacity C_{I}^D in* (14) *are equal for fixed channel matrix* $\mathbf{H}$, *that is,* $C_{\text{I}} = C_{\text{I}}^D$.

The proof of Lemma 1 is given in Appendix A.

Remark 1. For fixed noise covariance matrix eigenvalues and channel eigenvalues, the optimum transmit covariance matrix eigenvalues are given by the waterfilling solution. For fixed channel eigenvalues and transmit covariance matrix eigenvalues, the noise eigenvalues which minimize C_{I}^D can be easily found. Let ν denote the rank of the transmit covariance matrix $\mathbf{Q}$, that is, $\lambda_1(\mathbf{Q}) \geq \cdots \geq \lambda_\nu(\mathbf{Q}) > \lambda_{\nu+1}(\mathbf{Q}) = \cdots = \lambda_m(\mathbf{Q}) = 0$. We start with the Lagrangian of the minimization problem,

$$\begin{aligned} \mathcal{L}(\hat{\mathbf{\Lambda}}_Z,\mu) &= \sum_{i=1}^{\nu}\log\left(1+\frac{\lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})}{\lambda_i(\mathbf{Z})}\right)\\ &\quad+\mu\left(\sum_{l=1}^{\nu}\lambda_l(\mathbf{Z})-n_R\sigma_N^2\right)+\sum_{k=1}^{\nu}\xi_k\lambda_k(\mathbf{Z}). \end{aligned} \quad (16)$$

The Lagrangian multiplier ξ_k which ensure that the eigenvalues of the noise covariance matrix $\mathbf{Z}$ are greater than or equal to zero, are all equal to zero, because $\lambda_k(\mathbf{Z}) > 0$ for all $1 \leq k \leq \nu$. Otherwise, the mutual information would be infinity. Since the optimization problem is convex with respect to the noise eigenvalues, we have the necessary and sufficient Karush-Kuhn-Tucker (KKT) condition from (16),

$$\frac{\partial\mathcal{L}(\hat{\mathbf{\Lambda}}_Z,\mu)}{\partial\lambda_i(\mathbf{Z})} = -\frac{\lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})}{\lambda_i(\mathbf{Z})^2\left(1+\lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})/\lambda_i(\mathbf{Z})\right)}+\mu=0. \quad (17)$$

We solve (17) for $\lambda_i(\mathbf{Z})$ and obtain

$$\lambda_i^*(\mathbf{Z}) = \frac{\lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})}{2}\left(\sqrt{1+\frac{4}{\lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})\mu}}-1\right). \quad (18)$$

The Lagrangian multiplier μ has to be chosen so that $\sum_{i=1}^{n_R}\lambda_i^*(\mathbf{Z}) = n\sigma_N^2$.

In the second part of the proof, the worst-case noise covariance eigenvalues are further characterized. This is done using the KKT conditions for optimality of $\mathbf{Z}$ and $\mathbf{Q}$ in the following way. For fixed noise covariance matrix $\mathbf{Z}$, the optimal transmit covariance matrix $\mathbf{Q}^*$ is characterized by the corresponding KKT conditions. For fixed transmit covariance matrix $\mathbf{Q}$, the worst-case noise covariance matrix $\mathbf{Z}^*$ is characterized by the corresponding KKT conditions. The pair of covariance matrices $(\mathbf{Q}^*, \mathbf{Z}^*)$ is saddle point if and only if for fixed noise covariance matrix $\mathbf{Z}^*$, the KKT conditions are fulfilled by transmit covariance matrix $\mathbf{Q}^*$ and the other way round, if for fixed transmit covariance matrix $\mathbf{Q}^*$, the KKT conditions are fulfilled by the noise covariance matrix $\mathbf{Z}^*$. This approach results in the following Lemma 2.

Lemma 2. *The worst-case noise eigenvalues in* (18) *correspond to the optimal transmit covariance matrix eigenvalues which are given by the water-filling solution. Furthermore, the optimum transmit covariance matrix eigenvalues with SNR $\rho = P/(n_R\sigma_N^2)$ are given by*

$$\lambda_i^*(\mathbf{Q}) = P\cdot\left(\frac{\rho\lambda_i(\mathbf{H})/(1+\rho\lambda_i(\mathbf{H}))}{\sum_{k=1}^{\nu}(\rho\lambda_k(\mathbf{H}))/(1+\rho\lambda_k(\mathbf{H}))}\right). \quad (19)$$

The rank of the channel matrix $\mathbf{H}$ *is denoted by* ν. *The worst-case noise eigenvalues are then given by*

$$\lambda_i^*(\mathbf{Z}) = \frac{1}{\rho}\lambda_i^*(\mathbf{Q}). \quad (20)$$

This choice of transmit covariance matrix eigenvalues and noise covariance matrix eigenvalues fulfill the optimality conditions for the minimax problem.

The proof of Lemma 2 is given in Appendix B. Lemma 1 and Lemma 2 complete the proof. □

Remark 2. The optimal transmit strategy can be further characterized. For high SNR values, we obtain from (19)

$$\lim_{\sigma_N^2\to 0}\lambda_i^*(\mathbf{Q}) = \frac{P}{m} \quad (21)$$

as long as the channel matrix $\mathbf{H}$ has full rank m. For small SNR values, we obtain from (19)

$$\lim_{\sigma_N^2\to\infty}\lambda_i^*(\mathbf{Q}) = P\cdot\frac{\lambda_i(\mathbf{H})}{\sum_{k=1}^{\nu}\lambda_k(\mathbf{H})}. \quad (22)$$

In both cases, the rank of the transmit covariance matrix $\mathbf{Q}$ is given by $\nu = m$ as long as the channel has full rank. This characterization for high and low SNR values is illustrated in Section 6.3.

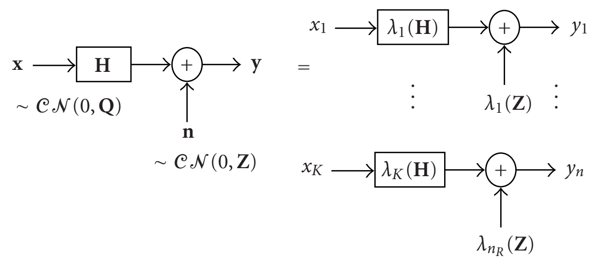

FIGURE 3: Worst-case noise directions: vector MIMO channel C_{II} and corresponding diagonalized orthogonal channels C_{II}^D.

4. WORST-CASE NOISE DIRECTIONS

We assume that the noise eigenvalues are fixed and ordered, that is, $\lambda_1(\mathbf{Z}) \geq \lambda_2(\mathbf{Z}) \geq \cdots \geq \lambda_{n_R}(\mathbf{Z})$. Here, we study the impact of the unitary matrix $\mathbf{U}_Z$. We write the set of unitary $n_R\times n_R$ matrices as $\mathcal{U}(n_R)$. We define the optimization problem as

$$C_{\text{II}} = \min_{\mathbf{W}\in\mathcal{U}(n_R)}\max_{\text{tr}(\mathbf{Q})\leq P}\log\frac{\det(\mathbf{W}\mathbf{\Lambda}_Z\mathbf{W}^H+\mathbf{HQH}^H)}{\det(\mathbf{W}\mathbf{\Lambda}_Z\mathbf{W}^H)}. \quad (23)$$

Furthermore, we define

$$C_{\text{II}}^D = \max_{\sum_{i=1}^{n_R}\lambda_i(\mathbf{Q})\leq P}\sum_{i=1}^{n_R}\log\left(1+\frac{\lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})}{\lambda_i(\mathbf{Z})}\right). \quad (24)$$

Obviously, the solution in (24) is the waterfilling solution.

The result in this section is that C_{II} and C_{II}^D are equal. In Figure 3, the correspondence between the closed-loop MIMO system with worst-case noise directions with noise covariance matrix $\mathbf{Z}$ and the system with parallel SISO channels $\lambda_1(\mathbf{H}),\ldots,\lambda_{n_R}(\mathbf{H})$ and noise variances $\lambda_1(\mathbf{Z}),\ldots,\lambda_{n_R}(\mathbf{Z})$ is shown.

In Scenario II, the worst-case directions deconstruct the MIMO channel into m orthogonal channels. Furthermore, the worst-case directions weight the noise covariance matrix eigenvalues in a way that the largest noise eigenvalues disturb the best channels. The capacity C_{II}^D in (24) has the nice property that it can be easily computed.

The worst-case noise unitary matrix is given by

$$\mathbf{U}_Z^{\min} = \mathbf{U}_H. \quad (25)$$

The best case noise unitary matrix (which achieves the upper bound in Theorem 1) is the permutation matrix times $\mathbf{U}_H$ which inverts the order of the noise eigenvalues.

We collect this result in the following theorem that states that the capacities in (23) and (24) are equal.

Theorem 3. *The capacity C_{II} in* (23) *and the capacity C_{II}^D in* (24) *are equal. The worst-case noise directions corresponds to the left eigenvectors of the channel matrix* $\mathbf{H}$.

Proof. First, we show that $C_{\mathrm{II}} \le C_{\mathrm{II}}^{D}$. We choose $\widehat{\mathbf{W}} = \mathbf{V}_H$. m is again defined as $m = \min(n_T, n_R)$. Then, it follows that

$$\begin{aligned} C_{\mathrm{II}} &\le \max_{\mathrm{tr}(\mathbf{Q})\le P} \log \frac{\det\left(\widehat{\mathbf{W}}\mathbf{\Lambda}_Z\widehat{\mathbf{W}}^H + \mathbf{V}_H\mathbf{\Lambda}_H^{1/2}\mathbf{Q}\mathbf{\Lambda}_H^{1/2}\mathbf{V}_H^H\right)}{\det\left(\widehat{\mathbf{W}}\mathbf{\Lambda}_Z\widehat{\mathbf{W}}^H\right)} \\ &= \max_{\mathrm{tr}(\mathbf{\Lambda}_Q)\le P} \sum_{i=1}^{m} \log\left(1 + \frac{\lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})}{\lambda_i(\mathbf{Z})}\right) = C_{\mathrm{II}}^{D}. \end{aligned} \tag{26}$$

Then, applying the Theorem 1 again, we show that $C_{\mathrm{II}} \ge C_{\mathrm{II}}^{D}$. We have

$$\begin{aligned} C_{\mathrm{II}} &\ge \max_{\mathrm{tr}(\mathbf{Q})\le P} \sum_{i=1}^{n_R} \log\left(1 + \frac{\lambda_i\left(\mathbf{\Lambda}_H^{1/2}\mathbf{Q}\mathbf{\Lambda}_H^{1/2}\right)}{\lambda_i(\mathbf{Z})}\right) \\ &= \max_{\mathrm{tr}(\mathbf{\Lambda}_Q)\le P} \sum_{i=1}^{m} \log\left(1 + \frac{\lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})}{\lambda_i(\mathbf{Z})}\right) = C_{\mathrm{II}}^{D}. \end{aligned} \tag{27}$$

From (26) and (27) follows $C_{\mathrm{II}} = C_{\mathrm{II}}^{D}$. This completes the proof. □

In addition to this, we will show in the following theorem that the optimization problem in (23) fulfills the saddle point property even though the set of unitary $n_R \times n_R$ matrices is not convex. The saddle point property is interesting for the game-theoretic interpretation of the minimax problem in (23). The order in which the two players, namely, the noise player and the transmit player, draw has no impact on the outcome.

Theorem 4. *The capacity*

$$C(\mathbf{W},\mathbf{Q}) = \frac{\log\left(\det\left(\mathbf{Z} + \mathbf{H}\mathbf{Q}\mathbf{H}^H\right)\right)}{\det \mathbf{Z}} \tag{28}$$

has the saddle point property, that is,

$$\min_{\mathbf{W}\in\mathcal{U}(n_R)} \max_{\mathrm{tr}(\mathbf{Q})\le P} C(\mathbf{W},\mathbf{Q}) = \max_{\mathrm{tr}(\mathbf{Q})\le P} \min_{\mathbf{W}\in\mathcal{U}(n_R)} C(\mathbf{W},\mathbf{Q}). \tag{29}$$

Proof. We know from the last theorem that

$$\begin{aligned} &\min_{\mathbf{W}\in\mathcal{U}(n_R)} \log \frac{\det\left(\mathbf{W}\mathbf{\Lambda}_Z\mathbf{W}^H + \mathbf{V}_H\mathbf{\Lambda}_H^{1/2}\mathbf{Q}\mathbf{\Lambda}_H^{1/2}\mathbf{V}_H^H\right)}{\det \mathbf{W}\mathbf{\Lambda}_Z\mathbf{W}^H} \\ &\qquad \ge \sum_{l=1}^{n_R} \log\left(1 + \frac{\lambda_l(\mathbf{H})\lambda_l(\mathbf{Q})}{\lambda_l(\mathbf{Z})}\right). \end{aligned} \tag{30}$$

We take the maximum over the transmit covariance matrix $\mathbf{Q}$ on the left-hand side (LHS) and right-hand side (RHS) of (30) and obtain with

$$\mathbf{\Phi} = \mathbf{W}\mathbf{\Lambda}_Z\mathbf{W}^H + \mathbf{V}_H\mathbf{\Lambda}_H^{1/2}\mathbf{Q}\mathbf{\Lambda}_H^{1/2}\mathbf{V}_H^H \tag{31}$$

the following inequality:

$$\begin{aligned} &\max_{\mathrm{tr}(\mathbf{Q})\le P} \min_{\mathbf{W}\in\mathcal{U}(n_R)} \log \frac{\det(\mathbf{\Phi})}{\det \mathbf{W}\mathbf{\Lambda}_Z\mathbf{W}^H} \\ &\qquad \ge \max_{\mathrm{tr}(\mathbf{Q})\le P} \sum_{l=1}^{n_R} \log\left(1 + \frac{\lambda_l(\mathbf{H})\lambda_l(\mathbf{Q})}{\lambda_l(\mathbf{Z})}\right). \end{aligned} \tag{32}$$

But the RHS of (32) is the minimax of $C(\mathbf{W},\mathbf{Q})$ and we have, therefore,

$$\begin{aligned} &\max_{\mathrm{tr}(\mathbf{Q})\le P} \min_{\mathbf{W}\in\mathcal{U}(n_R)} C(\mathbf{W},\mathbf{Q}) \\ &\qquad \ge \max_{\mathrm{tr}(\mathbf{Q})\le P} \sum_{l=1}^{n_R} \log\left(1 + \frac{\lambda_l(\mathbf{H})\lambda_l(\mathbf{Q})}{\lambda_l(\mathbf{Z})}\right) \\ &\qquad = \min_{\mathbf{W}\in\mathcal{U}(n_R)} \max_{\mathrm{tr}(\mathbf{Q})\le P} C(\mathbf{W},\mathbf{Q}). \end{aligned} \tag{33}$$

This completes the proof. □

Remark 3. In comparison with the worst-case noise system with trace constraint, the closed-loop MIMO system with worst-case noise directions looses again the transmitter cooperation, but still power allocation can be applied. The optimal power allocation is waterfilling with respect to the effective channel matrix eigenvalues $\lambda_i(\mathbf{H})/\lambda_i(\mathbf{Z})$.

In addition to this, for fixed noise covariance matrix eigenvalues, we can ask the other way round: what are the best-case noise directions? By applying Theorem 1 again, we have

$$\begin{aligned} &\max_{\mathbf{W}\in\mathcal{U}(n_R)} \log \frac{\det\left(\mathbf{W}\mathbf{\Lambda}_Z\mathbf{W}^H + \mathbf{H}\mathbf{Q}\mathbf{H}^H\right)}{\det\left(\mathbf{W}\mathbf{\Lambda}_Z\mathbf{W}^H\right)} \\ &\qquad = \sum_{i=1}^{m} \log\left(1 + \frac{\lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})}{\lambda_{m-i+1}(\mathbf{Z})}\right). \end{aligned} \tag{34}$$

The difference between the worst-case noise directions and the best case noise directions depends on the noise and channel eigenvalues $\lambda_i(\mathbf{H})$ and $\lambda_i(\mathbf{Z})$. In the next section, we illustrate the worst-case noise direction with an example.

4.1. Example of worst-case noise direction

In the following, we give an example of the worst-case noise direction. We assume two transmit and two receive antennas and fix the channel matrix

$$\mathbf{H} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \tag{35}$$

and noise eigenvalues with $\lambda_1(\mathbf{Z}) = 1.5\sigma_N^2$ and $\lambda_2(\mathbf{Z}) = 0.5\sigma_N^2$. We parameterize the unitary matrix $\mathbf{W}$ by

$$\mathbf{W} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}. \tag{36}$$

The effective channel $\widetilde{\mathbf{H}}$ is given by

$$\widetilde{\mathbf{H}} = \mathbf{\Lambda}_Z^{-1/2}\mathbf{W}^H(t)\mathbf{H}. \tag{37}$$

The eigenvalues of $\widetilde{\mathbf{H}}\widetilde{\mathbf{H}}^H$ in dependence of t can be computed in closed form. It can be shown that the maximum of the largest efficient channel eigenvalue occurs at $t = 0$ and is equal to $\lambda_1(\mathbf{H})/\lambda_2(\mathbf{Z})$. The minimum of the largest efficient channel matrix eigenvalue occurs at $t = \pi/2$ and equals $\lambda_1(\mathbf{H})/\lambda_1(\mathbf{Z})$. Analogously, we obtain for the smallest efficient

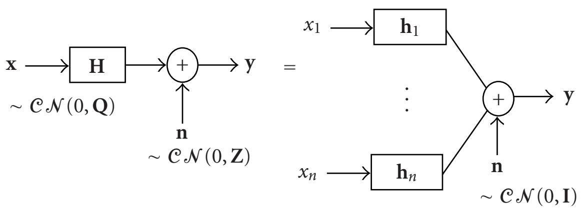

FIGURE 4: Worst-case colored noise: vector MIMO channel C_{III} and corresponding SIMO MAC channels C_{III}^D.

channel matrix eigenvalue the maximum at $t = \pi/2$ with $\lambda_2(\mathbf{H})/\lambda_2(\mathbf{Z})$ and the minimum at $t = 0$ with $\lambda_2(\mathbf{H})/\lambda_1(\mathbf{Z})$. The instantaneous mutual information can be computed in closed form as a function of t and it can be shown that the minimum mutual information occurs at $t = 0$ and the maximum at $t = \pi/2$. Therefore, we obtain for the worst-case noise directions an effective channel with eigenvalues $\lambda_1(\tilde{\mathbf{H}}) = 4/3$ and $\lambda_2(\tilde{\mathbf{H}}) = 2$. At 10 dB, waterfilling provides the optimal power allocation $\lambda_1(\mathbf{Q}) = 0.5125$ and $\lambda_2(\mathbf{Q}) = 0.4875$ which yields a capacity of 6.398 (bit/s/Hz).

5. WORST-CASE COLORED NOISE

In this scenario, the diagonal of the noise covariance matrix is equal to σ_N^2. We define the set of all noise covariance matrices with constant σ_N^2 entries on the diagonal as

$$\mathcal{Z} = \{\mathbf{Z} : \mathbf{Z} \succeq 0,\ \text{diag}(\mathbf{Z}) = [\sigma_N^2 \cdots \sigma_N^2]\}. \tag{38}$$

We define the capacity of the minimax problem as

$$C_{\text{III}} = \min_{\mathbf{Z}\in\mathcal{Z}} \max_{\text{tr}(\mathbf{Q})\le P} \log \frac{\det(\mathbf{Z} + \mathbf{HQH}^H)}{\det \mathbf{Z}}. \tag{39}$$

We define the capacity of the SIMO MAC as

$$C_{\text{III}}^D = \max_{\sum_{l=1}^n p_l \le P} \log\det\left(\mathbf{I} + \frac{1}{\sigma_N^2}\sum_{l=1}^{n_T} p_l \mathbf{h}_l \mathbf{h}_l^H\right). \tag{40}$$

The result of this section is that C_{III} and C_{III}^D are equal. We first describe the result. In Figure 4, the correspondence between the closed-loop MIMO system with worst-case colored noise and the SIMO MAC with white noise is shown. In Scenario III, there is cooperation at the transmit side only in terms of power control.

In Scenario III, we assume for convenience that $n_T = n_R$. The worst-case color of the noise reduces the achievable capacity of a MIMO system with n_T cooperating transmit antennas to n_T users who perform only power control. The achievable mutual information C_{III} for the MIMO channel with worst-case colored noise equals the sum capacity of the multiuser SIMO MAC. This fact has been used in [10] to derive an upper bound on the capacity region of the BC.

First, we will need the following lemma which explicitly states the reciprocity between uplink and downlink transmission.

Lemma 3. *The value of the following two optimization problems is equal:*

$$\begin{aligned}&\max_{\text{tr}(\mathbf{Q})\le P} \log\det(\mathbf{I} + \rho\mathbf{HQH}^H)\\ &\quad= \max_{\text{tr}(\mathbf{S})\le P} \log\det(\mathbf{I} + \rho\mathbf{H}^H\mathbf{SH}).\end{aligned} \tag{41}$$

Proof. The value of the two optimization problems in (41) does not depend on the left or right eigenvectors of $\mathbf{H}$, because $\log\det(\mathbf{I} + \mathbf{UAU}^H) = \log\det(\mathbf{I} + \mathbf{A})$ for unitary $\mathbf{U}$ and because $\text{tr}(\mathbf{UQU}^H) = \text{tr}(\mathbf{Q})$. Denote the rank of the $n_R \times n_T$ matrix $\mathbf{H}$ by ν. Furthermore, the value of the $\log\det(\mathbf{I}+\mathbf{X})$ at point $\mathbf{X} = \mathbf{0}$ is equal to zero, that is, $f(0) = 0$. As a result, the LHS of (41) is

$$\begin{aligned}&\max_{\text{tr}(\mathbf{Q})\le P} \log\det(\mathbf{I} + \mathbf{HQH}^H)\\ &\quad= \max_{\sum_{k=1}^m p_k \le P} \sum_{k=1}^m \log(1 + \lambda_k(\mathbf{H})p_k)\\ &\quad= \max_{\sum_{k=1}^\nu p_k \le P} \sum_{k=1}^\nu \log(1 + \lambda_k(\mathbf{H})p_k).\end{aligned} \tag{42}$$

The RHS of (41) is

$$\begin{aligned}&\max_{\text{tr}(\mathbf{S})\le P} \log\det(\mathbf{I} + \mathbf{H}^H\mathbf{SH})\\ &\quad= \max_{\sum_{k=1} s_k \le P} \sum_{k=1}^n \log(1 + \lambda_k(\mathbf{H})s_k)\\ &\quad= \max_{\sum_{k=1}^\nu s_k \le P} \sum_{k=1}^\nu \log(1 + \lambda_k(\mathbf{H})s_k)\end{aligned} \tag{43}$$

with eigenvalues $s_1, \ldots, s_n$ of $\mathbf{S}$. Equation (42) is equal to (43). This completes the proof. □

In the following theorem, we will show that the capacity C_{III} in (39) is equal to the capacity C_{III}^D in (40).

Theorem 5. *The capacity of the single user MIMO system with worst-case colored noise and perfect CSI at the transmitter as well as the receiver C_{III} is equal to the capacity of a multiuser SIMO MAC C_{III}^D with white noise.*

Proof. At first, for the capacity C_{III}, it holds that

$$C_{\text{III}} = \min_{\mathbf{Z}\in\mathcal{Z}} \max_{\text{tr}(\mathbf{Q})\le P} \log \frac{\det(\mathbf{Z} + \mathbf{HQH}^H)}{\det \mathbf{Z}} \tag{44a}$$

$$= \min_{\mathbf{Z}\in\mathcal{Z}} \max_{\text{tr}(\mathbf{Q})\le P} \log\det(\mathbf{I} + \mathbf{Z}^{-1/2}\mathbf{HQH}^H\mathbf{Z}^{-1/2}) \tag{44b}$$

$$= \min_{\mathbf{Z}\in\mathcal{Z}} \max_{\text{tr}(\mathbf{S})\le P} \log\det(\mathbf{I} + \mathbf{H}^H\mathbf{Z}^{-1/2}\mathbf{S}\mathbf{Z}^{-1/2}\mathbf{H}) \tag{44c}$$

$$= \min_{\mathbf{Z}\in\mathcal{Z}} \max_{\text{tr}(\mathbf{SZ})\le P} \log\det(\mathbf{I} + \mathbf{H}^H\mathbf{SH}). \tag{44d}$$

Step (44a) follows from the definition, in step (44b) the fact used is that $\mathbf{Z}$ is full rank because of its constraint. In step (44c), the point-to-point reciprocity in Lemma 3 is used and step (44d) is obvious. $\mathbf{S}$ is the transmit covariance matrix of the reciproc channel. We define the capacity in step (44d) as a function of the noise covariance matrix by

$$C_{IIIa}(\mathbf{Z}) = \max_{\mathbf{S}\succ 0,\, \mathrm{tr}(\mathbf{SZ})\le P} \log\det\left(\mathbf{I}+\mathbf{H}^H\mathbf{S}\mathbf{H}\right). \tag{45}$$

It is obvious that the feasible input to the MAC capacity in C_{III}^D in (40) fulfills the trace constraint in (45), because $\mathbf{Z}_{ii} \le \sigma_N^2$. Therefore, we have $C_{\mathrm{III}}^D \le C_{\mathrm{III}} \le C_{IIIa}(\mathbf{Z})$. Now, we find a $\mathbf{Z}^*$ with diagonal elements $\le \sigma_N^2$ such that $C_{IIIa}(\mathbf{Z}^*) = C_{\mathrm{III}}^D$, that is, a $\mathbf{Z}^*$ such that the optimal input $\mathbf{S}^* = \mathrm{diag}(p_1,\ldots,p_n)$ for the MAC is also optimal for the C_{III} in (39). The Lagrangian for the optimization in (45) is given by

$$L(\mathbf{S},\lambda) = \log\det\left(\mathbf{I}+\mathbf{H}^H\mathbf{S}\mathbf{H}\right) - \lambda\left(\mathrm{tr}(\mathbf{SZ}) - P\right) \tag{46}$$

with Lagrangian multiplier λ for the power constraint. A sufficient condition for optimality of some $\mathbf{S}^*$ is given by the KKT conditions:

$$\begin{gathered}\mathrm{tr}(\mathbf{SZ}) \le P,\\ \left.\frac{\partial L(\mathbf{S},\lambda)}{\partial \mathbf{S}}\right|_{\mathbf{S}=\mathbf{S}^*} = 0\end{gathered} \tag{47}$$

for some $\lambda > 0$. This condition is obviously satisfied by

$$\mathbf{Z}^* = \frac{1}{\lambda}\mathbf{H}\left(\mathbf{I}+\mathbf{H}^H\mathbf{S}^*\mathbf{H}\right)^{-1}\mathbf{H}^H. \tag{48}$$

To satisfy the trace constraint, λ is derived from the KKT condition for the MAC channel in (40) as

$$\mathbf{h}_i\left(\mathbf{I}+\mathbf{H}^H\mathbf{S}^*\mathbf{H}\right)^{-1}\mathbf{h}_i^H = \lambda \tag{49}$$

for all $p_i > 0$. The λ from (49) in (48) ensures that the trace constraint $\mathrm{tr}(\mathbf{ZS}) \le P$ is satisfied. This completes the proof. □

Remark 4. Our proof of Theorem 5 goes in similar lines like the derivation in [24, Section III.C].

The closed-loop MIMO system with worst-case colored noise looses the cooperation between the transmit antennas. However, power allocation is still possible. Therefore, we arrive at the SIMO MAC with power allocation. This decomposition of the MIMO system with worst-case colored noise into the SIMO MAC with white noise has been used in the literature in various contexts. The Sato bound in [10] is an upper bound for the capacity region of a BC. The region is upper bounded by the capacity of the cooperative (at transmit side) single-user system with worst-case colored noise. The sum capacity of the multiuser system and the capacity of the single user systems are equal. This way Theorem 5 can be proven, too [11]. The worst-case capacity of Gaussian vector BC is further analyzed in [9].

5.1. Example for worst-case colored noise

In the following, we give two examples for the computation of the worst-case colored noise saddle point by solving the simple SIMO MAC problem. In the first example, both users are supported. In the second example, only one mobile user is supported.

5.1.1. Example A

Let the channel matrix be given by

$$\mathbf{H} = \begin{pmatrix}0.1 & 0.5\\ 0.8 & 0.2\end{pmatrix} \tag{50}$$

with transmit power constraint $P = 10$ and noise power $\sigma_N^2 = 1$. We apply the following steps.

(1) We compute the optimal power allocation for the SIMO MAC by MAXDET [25]

$$\mathbf{S}^* = \mathrm{diag}\left([3.5457, 6.4543]\right). \tag{51}$$

The Lagrangian multiplier is $\lambda_1 = \lambda_2 = 0.124$.

(2) We compute the worst-case colored noise in (48) and the corresponding KKT condition yields

$$\mathbf{Z}^* = \begin{pmatrix}1 & 0.151\\ 0.151 & 1\end{pmatrix}. \tag{52}$$

(3) Waterfilling with respect to the effective channel provides the optimal MIMO transmission strategy:

$$\mathbf{Q}^* = \begin{pmatrix}6.2675 & 1.0151\\ 1.0151 & 3.7325\end{pmatrix}. \tag{53}$$

Next, we test this result by computing the sum capacity of the SIMO MAC with $\mathbf{S}^*$ and by computing the capacity of the MIMO channel with worst-case colored noise $\mathbf{Z}^*$ and transmit strategy $\mathbf{Q}^*$:

$$\begin{aligned}C_{\mathrm{III}}^D &= \log\det\left(\mathbf{I}+\mathbf{H}^H\mathbf{S}^*\mathbf{H}\right) = 3.2653,\\ C_{\mathrm{III}} &= \log\det\left(\mathbf{Z}^*+\mathbf{H}\mathbf{Q}^*\mathbf{H}^H\right) - \log\det\mathbf{Z} = 3.2653.\end{aligned} \tag{54}$$

We have

$$C_{\mathrm{III}}^D = C_{\mathrm{III}}. \tag{55}$$

5.1.2. Example B

We consider the same channel as in example A and the same noise variance. We choose the transmit power $P = 1$.

(1) The optimal power allocation for the SIMO MAC is given by

$$\mathbf{S}^* = \mathrm{diag}\left([0, 1]\right) \tag{56}$$

and the Lagrangian multiplier is $\lambda_1 = 0.2407 < 0.4048 = \lambda_2$.

(2) The corresponding worst-case colored noise for the single-user MIMO system is

$$\mathbf{Z}^* = \begin{pmatrix} 1(0.5946) & 0.2647 \\ 0.2647 & 1 \end{pmatrix}. \quad (57)$$

Here, the entry $(1, 1)$ in $\mathbf{Z}^*$ was filled up from 0.5946 to 1. This does not change the optimal $\mathbf{Q}^*$ nor the value of the minmax problem (see [19]).

(3) The waterfilling solution yields

$$\mathbf{Q}^* = \begin{pmatrix} 0.9412 & 0.2353 \\ 0.2353 & 0.0588 \end{pmatrix}. \quad (58)$$

As in the previous example, we have

$$C_{\text{III}}^D = 0.7485 = C_{\text{III}}. \quad (59)$$

6. INTERPRETATION, ILLUSTRATION, AND DISCUSSION

In all three scenarios, it is shown that the achievable capacity with optimal transmit covariance matrix under worst-case noise (with different constraints) equals the achievable capacity without transmit cooperation and i.i.d. noise with scaled identity covariance matrix. The differences and commonness between CSI and cooperation are further discussed in the following.

6.1. Discussion of results

We have considered a scenario in which the transmitter has perfect CSI $\mathbf{H}$ and further knowledge of the interference covariance matrix $\mathbf{Z}$. Obviously, both the capacity C_x and the optimal transmit covariance matrix $\mathbf{Q}$ are a function of the noise covariance matrix $\mathbf{Z}$. In all three noise scenarios, we have searched for the global minimum of C_x with respect to $\mathbf{Z}$. From a multiuser point of view, this corresponds to the question "what is the worst-case interference that limits the capacity of my link?". Closely related are the questions "how much information rate can be guaranteed even in worst-case noise?" and "what is lost due to worst-case noise?". In general MIMO systems, the transmit antennas can cooperate, that is, some kind of beamforming can be applied. A necessary condition for cooperation is some kind of CSI at the transmit side. This means that there are different stages of transmit operation. Without CSI, there is no cooperation.

In order to summarize the results and answer of the three questions from the last paragraph, the following list is proposed.

(i) *Worst-case noise with trace constraint*. Regarding the channel capacity, a MIMO system with perfect CSI about $\mathbf{H}$ and about interference $\mathbf{Z}$ at both sides of the link equals a MIMO system without CSI at the transmitter and with slightly amplified white noise. Due to worst-case noise, the transmitter loses its CSI and hence its cooperation.

(ii) *Worst-case noise directions*. In this scenario, the MIMO system with perfect CSI about $\mathbf{H}$ and $\mathbf{Z}$ transforms under worst-case noise directions with fixed noise covariance matrix eigenvalues to a system with parallel fading channels with effective channel gains $\lambda_i(\mathbf{H})/\lambda_i(\mathbf{Z})$. At the transmitter, CSI and cooperation were available and necessary to diagonalize the channel. The last optimization step is power allocation according to waterfilling against the effective channel.

(iii) *Worst-case colored noise*. Regarding the channel capacity, a MIMO system with perfect CSI about $\mathbf{H}$ and $\mathbf{Z}$ under worst-case colored noise equals a SIMO MAC with CSI at the transmitters and white noise. In this scenario, the CSI is still available at the transmit antenna. In exchange, the cooperation at the transmit side is lost.

6.2. Comparison of worst-case noise capacities

In Scenario I, the worst-case noise has the same directions as in Scenario II. Additionally, the eigenvalues of the noise covariance matrix are chosen to minimize the mutual information. The optimal noise covariance matrix eigenvalues are explicitly given in (18). The minimax problem in (12) fulfills the saddle point property. Therefore, we have, for all $\mathbf{Q}$ and $\mathbf{Z}$ and with optimal pair $(\mathbf{Q}^*, \mathbf{Z}^*)$,

$$F(\mathbf{Q}, \mathbf{Z}^*) \le F(\mathbf{Q}^*, \mathbf{Z}^*) \le F(\mathbf{Q}^*, \mathbf{Z}). \quad (60)$$

For fixed $\mathbf{Z}$, the optimal $\mathbf{Q}$ is the waterfilling solution and for fixed $\mathbf{Q}$ the noise covariance matrix which minimizes the mutual information is given by (18). In general, we have, for the eigenvectors of the optimal $\mathbf{Q}^*$ and $\mathbf{Z}^*$,

$$\begin{aligned} \mathbf{Q}^* &= \mathbf{V}_H \mathbf{\Lambda}_{\mathbf{Q}^*} \mathbf{V}_H^H, \\ \mathbf{Z}^* &= \mathbf{U}_H \mathbf{\Lambda}_{\mathbf{Z}^*} \mathbf{U}_H^H. \end{aligned} \quad (61)$$

In Scenario I, the set of admissible noise covariance matrices is larger than in Scenario II. The additional choice of eigenvalue distribution reveals this. Therefore, the capacity C_{I} is smaller than or equal to C_{II}.

The capacity C_{I} is smaller than or equal to C_{III}. This follows from the fact that the set of feasible noise covariance matrices in Scenario I and Scenario II is larger than the set of feasible noise covariance matrices in Scenario III. We obtain

$$\begin{aligned} C_{\text{I}} &\le C_{\text{II}}, \\ C_{\text{I}} &\le C_{\text{III}}. \end{aligned} \quad (62)$$

Unfortunately, we cannot compare the capacities C_{II} and C_{III}, because the set of noise covariance matrices in Scenario II is not a subset of the set in Scenario III and vice versa.

6.3. Illustration of worst-case noise with trace constraint

In Figure 5, the worst-case noise covariance matrix eigenvalue $\lambda_1(\mathbf{Z})$ is shown over the SNR and the channel matrix eigenvalue $\lambda_1(\mathbf{H})$. The noise, channel, and transmit covariance matrices are sum-normalized, that is, $\lambda_1(\mathbf{Z}) + \lambda_2(\mathbf{Z}) = 1$, $\lambda_1(\mathbf{H}) + \lambda_2(\mathbf{H}) = 1$, and $\lambda_1(\mathbf{Q}) + \lambda_2(\mathbf{Q}) = 1$, respectively.

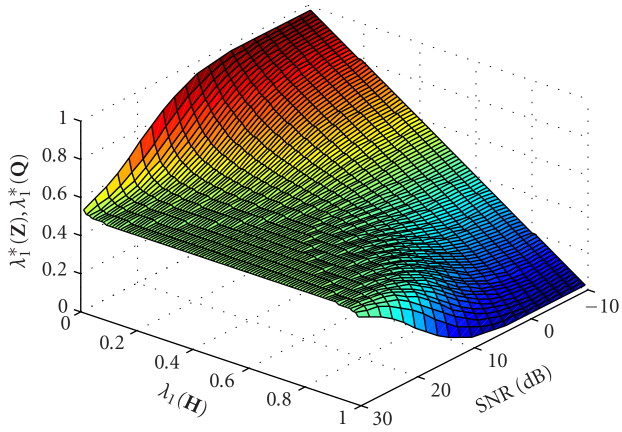

FIGURE 5: MIMO 2 × 2. Normalized worst-case noise covariance matrix eigenvalue $\lambda_1(\mathbf{Z})$ and normalized optimal transmit covariance matrix eigenvalue $\lambda_1(\mathbf{Q})$ over SNR (dB) and channel matrix eigenvalue $\lambda_1(\mathbf{H})$.

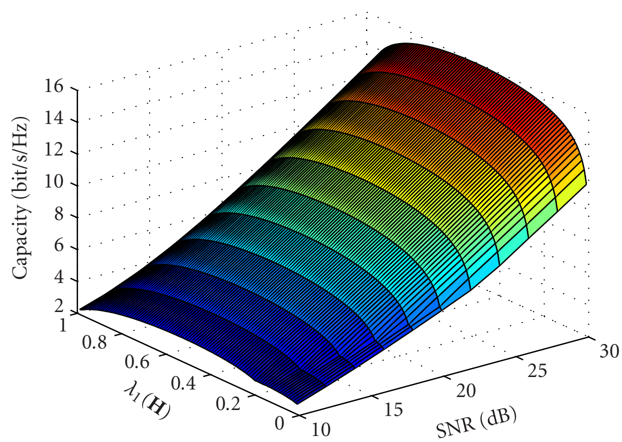

FIGURE 6: Mutual information using $\lambda_1(\mathbf{Q})$ and $\lambda_1(\mathbf{Z})$ from Figure 5 over SNR (dB) and channel matrix eigenvalue $\lambda_1(\mathbf{H})$.

In Figure 5, we observe that for small SNR values the worst-case noise eigenvalues correspond to the channel matrix eigenvalues as predicted by (22). For high SNR values, the worst-case noise eigenvalues have equal power. In Figure 6, we show the mutual information which is achieved by $\lambda_1(\mathbf{Z})$ and $\lambda_1(\mathbf{Q})$ from Figure 5.

7. CONCLUSION

In this work, the instantaneous capacity of a MIMO system with worst-case noise was studied. The three different noise scenarios lead to different noise constraints and different worst-case noise capacities. We studied a trace constraint, fixed noise eigenvalues but free noise directions, and fixed diagonal entries in noise covariance matrix $\mathbf{Z}$ but free color of noise. In all three scenarios, the MIMO system lost its ability to cooperate at the transmitter side. In the first case with worst-case noise and trace constraint, the CSI at the transmitter dropped away. In scenario II and scenario III, the eigendirections of the transmit covariance matrices were canceled by the worst-case noise direction or color and only power allocation could be performed at the transmitter.

APPENDICES

A. PROOF OF LEMMA 1

At first, we show that $C_{\mathrm{I}} \le C_{\mathrm{I}}^{D}$. We have

$$\begin{aligned}
\max_{\operatorname{tr}(\mathbf{Q})\le P} \log \frac{\det(\mathbf{Z}+\mathbf{H}\mathbf{Q}\mathbf{H}^H)}{\det \mathbf{Z}}
&= \max_{\operatorname{tr}(\mathbf{Q})\le P} \log \frac{\det(\mathbf{U}_H^H\mathbf{Z}\mathbf{U}_H + \mathbf{\Lambda}_H^{1/2}\mathbf{V}_H^H\mathbf{Q}\mathbf{V}_H\mathbf{\Lambda}_H^{1/2})}{\det \mathbf{U}_H^H\mathbf{Z}\mathbf{U}_H} \\
&= \max_{\operatorname{tr}(\mathbf{V}_H\mathbf{Q}\mathbf{V}_H^H)\le P} \log \frac{\det(\mathbf{U}_H^H\mathbf{Z}\mathbf{U}_H + \mathbf{\Lambda}_H^{1/2}\mathbf{Q}\mathbf{\Lambda}_H^{1/2})}{\det \mathbf{U}_H^H\mathbf{Z}\mathbf{U}_H} \\
&= \max_{\operatorname{tr}(\mathbf{Q})\le P} \log \frac{\det(\mathbf{U}_H^H\mathbf{Z}\mathbf{U}_H + \mathbf{\Lambda}_H^{1/2}\mathbf{Q}\mathbf{\Lambda}_H^{1/2})}{\det \mathbf{U}_H^H\mathbf{Z}\mathbf{U}_H}.
\end{aligned} \tag{A.1}$$

Now, we choose $\hat{\mathbf{Z}} = \mathbf{U}_H\mathbf{\Lambda}_Z\mathbf{U}_H^H$ fixed, let the eigenvalues vary, and obtain an upper bound on C_{I}:

$$\begin{aligned}
C_{\mathrm{I}} &= \min_{\operatorname{tr}(\mathbf{Z})\le\sigma_N^2 n}\ \max_{\operatorname{tr}(\mathbf{Q})\le P} \log \frac{\det(\mathbf{Z}+\mathbf{H}\mathbf{Q}\mathbf{H}^H)}{\det \mathbf{Z}} \\
&\le \min_{\sum_{k=1}^{n}\lambda_k(\mathbf{Z})\le n\sigma^2}\ \max_{\operatorname{tr}(\mathbf{Q})\le P} \log \frac{\det(\mathbf{\Lambda}_Z + \mathbf{\Lambda}_H^{1/2}\mathbf{Q}\mathbf{\Lambda}_H^{1/2})}{\det \mathbf{\Lambda}_Z} \\
&= \min_{\sum_{k=1}^{n}\lambda_k(\mathbf{Z})\le n\sigma^2}\ \max_{\operatorname{tr}(\mathbf{\Lambda}_Q)\le P} \log \frac{\det(\mathbf{\Lambda}_Z + \mathbf{\Lambda}_H\mathbf{\Lambda}_Q)}{\det \mathbf{\Lambda}_Z} = C_{\mathrm{I}}^{D}.
\end{aligned} \tag{A.2}$$

The last equality in (A.2) follows from the fact that the optimal eigenvectors of $\mathbf{Q}$ are equal to identity. The LHS of (A.2) does not depend on $\mathbf{\Lambda}_Z$. Next, we show using Theorem 1 that $C_{\mathrm{I}} \ge C_{\mathrm{I}}^{D}$. With Theorem 1, we have

$$\begin{aligned}
\log \frac{\det(\mathbf{Z}+\mathbf{H}\mathbf{Q}\mathbf{H}^H)}{\det \mathbf{Z}}
\ge \sum_{i=1}^{m} \log \frac{\lambda_i(\mathbf{Z}) + \lambda_i(\mathbf{H}\mathbf{Q}\mathbf{H}^H)}{\lambda_i(\mathbf{Z})}.
\end{aligned} \tag{A.3}$$

The maximum over $\mathbf{Q}$ of the term in (A.3) is greater than or equal to the term with the choice of $\mathbf{U}_Q = \mathbf{U}_H^H$, that is,

$$\begin{aligned}
\max_{\operatorname{tr}(\mathbf{Q})\le P} \log \frac{\det(\mathbf{Z}+\mathbf{H}\mathbf{Q}\mathbf{H}^H)}{\det \mathbf{Z}}
\ge \sum_{i=1}^{m} \log \frac{\lambda_i(\mathbf{Z}) + \lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})}{\lambda_i(\mathbf{Z})}.
\end{aligned} \tag{A.4}$$

Inequality (A.4) is valid for all $\mathbf{Z}$. Therefore, we have

$$\begin{aligned}
&\min_{\operatorname{tr}(\mathbf{Z})\le n_R\sigma_N^2}\ \max_{\operatorname{tr}(\mathbf{Q})\le P} \log \frac{\det(\mathbf{Z}+\mathbf{H}\mathbf{Q}\mathbf{H}^H)}{\det \mathbf{Z}} \\
&\quad\ge \min_{\operatorname{tr}(\mathbf{Z})\le n_R\sigma_N^2}\ \max_{\operatorname{tr}(\mathbf{Q})\le P} \sum_{i=1}^{m} \log \frac{\lambda_i(\mathbf{Z}) + \lambda_i(\mathbf{H})\lambda_i(\mathbf{Q})}{\lambda_i(\mathbf{Z})}.
\end{aligned} \tag{A.5}$$

From (A.5), it follows that

$$C_{\mathrm{I}} \geq C_{\mathrm{I}}^{D}. \tag{A.6}$$

From (A.2) and (A.6) follows $C_{\mathrm{I}} = C_{\mathrm{I}}^{D}$. This completes the proof.

B. PROOF OF LEMMA 2

The proof of Lemma 2 can be described as follows. Choose the eigenvalues of the noise covariance matrix to be equal to the weighted eigenvalues of the transmit covariance matrix, that is, $\lambda_i(\mathbf{Q}) = \rho\lambda_i(\mathbf{Z})$, and show that the optimality conditions for the minimization with respect to $\mathbf{Z}$ is fulfilled. Choose the eigenvalues of the transmit covariance matrix to be equal to the weighted eigenvalues of the noise covariance matrix and show that the optimality conditions for the maximization with respect to $\mathbf{Q}$ are fulfilled.

We denote the optimal transmit covariance matrix eigenvalues by $\lambda_i^*(\mathbf{Q})$ and the worst-case noise covariance matrix eigenvalues by $\lambda_i^*(\mathbf{Z})$. The waterfilling solution of the transmit covariance matrix eigenvalues is given for all $\lambda_i^*(\mathbf{Q}) > 0$ as

$$\lambda_i^*(\mathbf{Q}) = \xi - \frac{\lambda_i^*(\mathbf{Z})}{\lambda_i(\mathbf{H})} \tag{B.1}$$

with $\xi > 0$. We show that the choice $\lambda_i^*(\mathbf{Z}) = (1/\rho)\lambda_i^*(\mathbf{Q})$ fulfills both optimality conditions (18) and (B.1), simultaneously. This result is derived by computing the Lagrangian multiplier for (B.1) and (18) and showing that $\xi = 1/\mu$. From (18), we have for $\lambda_i^*(\mathbf{Q}) = \rho\lambda_i^*(\mathbf{Z})$,

$$\begin{aligned}\lambda_i^*(\mathbf{Z}) = {} & \frac{1}{2}\rho\lambda_i^*(\mathbf{Z})\lambda_i(\mathbf{H}) \\ & \cdot \left(\sqrt{1 + \frac{4}{\rho\lambda_i(\mathbf{H})\lambda_i^*(\mathbf{Z})\mu}} - 1\right).\end{aligned} \tag{B.2}$$

Solving (B.2) for μ yields

$$\frac{1}{\mu} = \lambda_i^*(\mathbf{Z})\left(\frac{1}{\rho\lambda_i(\mathbf{H})} + 1\right). \tag{B.3}$$

From (B.1), we have for $\lambda_i^*(\mathbf{Z}) = (1/\rho)\lambda_i^*(\mathbf{Q})$,

$$\lambda_i^*(\mathbf{Q}) = \xi - \frac{\lambda_i^*(\mathbf{Q})}{\rho\lambda_i(\mathbf{H})}. \tag{B.4}$$

Solving (B.4) for ξ gives

$$\xi = \lambda_i^*(\mathbf{Q})\left(\frac{1}{\rho\lambda_i(\mathbf{H})} + 1\right) = \frac{1}{\mu}. \tag{B.5}$$

Equation (B.5) connects the Lagrangian multiplier for the transmit covariance matrix optimization in (B.1) and the worst-case noise optimization in (18). This shows that $\lambda_i^*(\mathbf{Q}) = \rho\lambda_i^*(\mathbf{Z})$ solves (B.1) and (18).

The closed-form expression for $\lambda_i^*(\mathbf{Q})$ in (19) is easily obtained from the power constraint:

$$\sum_{k=1}^{m} \lambda_i^*(\mathbf{Q}) = P. \tag{B.6}$$

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