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A General Theory for SIR Balancing

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We study the problem of maximizing the minimum signal-to-interference ratio (SIR) in a multiuser system with an adaptive receive strategy. The interference of each user is modelled by an axiomatic framework, which reflects the interaction between the propagation channel, the power allocation, and the receive strategy used for interference mitigation. Assuming that there is a one-to-one mapping between the QoS and the signal-to-interference ratio (SIR), the feasible QoS region is completely characterized by the max-min SIR balancing problem. In the first part of the paper, we derive fundamental properties of this problem for the most general case, when interference is modelled with an axiomatic framework. In the second part, we show more specific properties for interference functions based on a nonnegative coupling matrix. The principal aim of this paper is to provide a deeper understanding of the interaction between power allocation and interference mitigation strategies. We show how the proposed axiomatic approach is related to the matrix-based theory.

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1. INTRODUCTION

A fundamental problem in wireless multiuser communications is the mitigation and control of interference. This is especially true for densely populated networks, where many mobile terminals share the same resource, so interference can have a large impact on the achievable quality of service (QoS).

Orthogonalization of the resources, like the TDMA or the FDMA, is neither always possible nor desirable. The available bandwidth is often best exploited by letting signals interfere with each other in a controlled way, for example, by using multiuser detection strategies (see, e.g., [1]). Also, orthogonality may be lost because of system imperfections and the effects of the time-varying multipath channel. So interference can be seen as the main hurdle in achieving a high per-user throughput in heavily loaded multiuser networks, as will be required in the future.

The traditional approach to wireless networking is the assumption of point-to-point communication links, which can be optimized independently. This strategy need not be a good choice for an interference-limited network, where the choice of one users' transmission strategy determines how much interference is received by another user. A link-centric optimization strategy would easily result in a competitive situation, where each user tries to counterbalance the interference by increasing its own power level, which in turn can cause even more interference to the overall system. This motivates joint optimization strategies, taking into account the interference coupling between the users.

1.1. The QoS feasible region

Joint optimization of multiple communication links can be performed with respect to different design goals. Possible strategies are restricted to the *QoS feasible region*, that is, the set of jointly achievable QoS. This region depends on the underlying channel properties and the chosen *receive strategy*. The term "receive strategy," which will be specified later, stands for the possible use of adaptive techniques, like linear interference filtering, interference cancellation, scheduling, and so forth.

Consider that K communication links with transmission powers

$$\mathbf{p} = \left[p_1, \dots, p_K\right]^T \in \mathbb{R}_+^K \text{ (nonnegative orthant).} \qquad (1)$$

Since all users are coupled by the interference, the signal-to-interference ratio (SIR) of each user is a function of *all* powers, that is,

$$\mathrm{SIR}_k(\mathbf{p}) = \frac{\mathbf{p}_k}{\mathcal{I}_k(\mathbf{p})}, \quad k \in \{1,2,\ldots,K\}, \tag{2}$$

where $\mathcal{I}_k(\mathbf{p})$ denotes the interference power of the kth user. Note that $\mathcal{I}_k(\mathbf{p})$ can possibly incorporate an adaptive receive strategy, so the dependency on the noise p can be nonlinear (see discussion in Section 2). Also, $\mathcal{I}_k(\mathbf{p})$ possibly includes a noise power component. However, the assumption of noise is not required for the following results.

The SIR is an important performance measure, which is often linked with the QoS by a strictly monotone function ϕ:

$$\mathrm{QoS}_k(\mathbf{p}) = \phi(\mathrm{SIR}_k(\mathbf{p})), \quad 1 \le k \le K. \tag{3}$$

Examples are the BER slope for α-fold diversity: $\phi(\mathrm{SIR}_k) = 1/\mathrm{SIR}_k^{\alpha}$, or the information-theoretical capacity: $\phi(\mathrm{SIR}_k) = \log(1+\mathrm{SIR}_k)$ for Gaussian signalling (see, e.g., [2, 3] for a more detailed discussion).

Because the mapping (3) is one-to-one, we need not study the QoS region directly. It is sufficient to study the SIR feasible region

$$\mathcal{S} = \{[\mathrm{SIR}_1(\mathbf{p}),\ldots,\mathrm{SIR}_K(\mathbf{p})] : \mathbf{p} \ge 0\}. \tag{4}$$

All results immediately transfer to the QoS feasible region $\mathcal{Q} = \phi(\mathcal{S})$.

The literature has many examples of optimization over the SIR region $\mathcal{S}$. The actual problem structure depends very much on the definition of the interference function $\mathcal{I}_k(\mathbf{p})$. In the following, we give a brief overview.

1.2. Related work

A widely used interference model (see, e.g., [4] and the references therein) is $\mathcal{I}_k(\mathbf{p}) = [\mathbf{\Psi}\mathbf{p}]_k$, where $\mathbf{\Psi}$ is a positive matrix which contains the crosstalk coefficients. The coefficient $\mathbf{\Psi}_{kl}$ determines the power crosstalk of the lth transmitter to the kth receiver. So the vector $\mathbf{\Psi}\mathbf{p}$ contains the total interference powers experienced by all K users, and $[\mathbf{\Psi}\mathbf{p}]_k$ is the kth component. For this model, the feasible region $\mathcal{S}$ is fully characterized by the maximum eigenvalue of the matrix $\mathbf{\Psi}$. This is a longstanding result from power-control theory, which is based on the idea of balancing the SIRs of all links on a common maximum level [5, 6]. Geometrical properties of $\mathcal{S}$, like convexity, were studied in [2, 3, 7].

Another model is the power-control framework of Yates [8], where $\mathcal{I}_k(\mathbf{p})$ is defined by a system of axioms capturing some basic properties of the interference functions. One important aspect of this model is the property $\alpha\mathcal{I}_k(\mathbf{p}) > \mathcal{I}_k(\alpha\mathbf{p})$ for $\alpha > 1$. This is not fulfilled by the model $\mathcal{I}_k(\mathbf{p}) = [\mathbf{\Psi}\mathbf{p}]_k$, thus the axiomatic framework [8] is not suitable for studying the above SIR balancing problem. Instead, it is very useful for deriving algorithmic solutions in the presence of noise. As an example, think of the interference function $\mathcal{I}_k(\mathbf{p}) = [\mathbf{\Psi}\mathbf{p}]_k + \sigma^2$, where σ^2 is the receiver noise power. If SIR targets $\gamma_1,\ldots,\gamma_K$ are feasible, then it was shown in [8] that the iteration $\mathbf{p}^{(n)} = \gamma_k\mathcal{I}_k(\mathbf{p}^{(n-1)})$ converges to a power vector $\hat{\mathbf{p}}$, which is the unique optimizer of the *sum-power minimization problem* $\min_{\mathbf{p}>0}\sum_l \mathbf{p}_l$ subject to $\mathbf{p}_k/\mathcal{I}_k(\mathbf{p}) \ge \gamma_k$ for all $k = 1,2,\ldots,K$. Note that this problem formulation is only meaningful in the presence of noise. For the noiseless model $\mathcal{I}_k(\mathbf{p}) = [\mathbf{\Psi}\mathbf{p}]_k$, the SIR is invariant with respect to a scaling of the power vector, that is, $\alpha\mathcal{I}_k(\mathbf{p}) = \mathcal{I}_k(\alpha\mathbf{p})$. This means that the power allocation $\mathbf{p}$ can be arbitrarily scaled, so the sum power $\|\mathbf{p}\|_1$ does not matter.

The axiomatic approach is attractive since it allows to study many power control problems in a common analytical framework. One example is the problem of joint power control and base station assignment. In [9, 10], it was proposed to define $\mathcal{I}_k(\mathbf{p})$ such that it includes an adaptive assignment of base stations. This approach allows for the development of efficient iterative algorithms for a problem which would otherwise be difficult to handle.

A further example is the joint optimization of beamforming and power control in the presence of noise [11–16], where the interference function takes on the form $\mathcal{I}_k(\mathbf{p}) = \min_{\boldsymbol{U}}[\mathbf{\Psi}(\boldsymbol{U})\mathbf{p}]_k + \sigma^2$. Here the link gain matrix $\mathbf{\Psi}$ depends on the choice beamforming filters, collected in a matrix $\boldsymbol{U}$. Again, this interference model can be shown to fulfill the axioms in [8], so we can iteratively find the power allocation which solves the sum-power minimization problem.

Another line of research is the joint optimization of beamforming and power control in the absence of noise [17–20], where the interference function has the special form $\mathcal{I}_k(\mathbf{p}) = \min_{\boldsymbol{U}}[\mathbf{\Psi}(\boldsymbol{U})\mathbf{p}]_k$. Although this seems to be a special case of the above model with $\sigma^2 = 0$, the absence of noise can drastically change the behavior of $\mathcal{I}_k(\mathbf{p})$. In particular, it is no longer possible to use the axiomatic model [8] for analysis. Also, the power minimization strategy is not a reasonable problem formulation, since the SIR is invariant with respect to a scaling of $\mathbf{p}$. Thus, research in [17–20] is mainly focused on the max-min SIR balancing problem, which can be recast as an optimization of the spectral radius $\rho(\mathbf{\Psi}(\boldsymbol{U}))$. Algorithmic solutions were derived under the assumption that $\mathbf{\Psi}(\boldsymbol{U})$ is always *irreducible*, which basically means that all users are coupled by interference. An overview on beamforming in a network context can be found in [21].

1.3. Motivation and contribution of the paper

One lesson from the literature is the importance of an axiomatic approach, which is specific enough to capture underlying effects of interference coupling, but general enough to allow the application to a wide range of problems in wireless communications. In particular, it is important to include possible "receive strategies," which will play an increasingly important role for future systems, where optimization is performed over functionalities of different layers. Examples of such a joint optimization are the aforementioned joint power control and channel assignment, which are closely related to scheduling issues. Another example is the joint optimization of physical layer interference mitigation and power control. The axiomatic theory is also very useful in including additional power constraints, as was shown in [8].

However, the axiomatic model in [8] only holds under the assumption of receiver noise. While this assumption seems to be perfectly justified, it also can cause problems. Namely, it does not allow to study the SIR balancing problem, as discussed above. But the noiseless case plays a significant role for the characterization of the QoS feasible region, which is the union of all power-constrained regions. This overall QoS region is only limited by the effects of interference. In order to derive necessary and sufficient conditions for feasibility of certain QoS targets, it is thus necessary to study the SIR balancing problem.

So one main goal of this paper is to derive a general axiomatic theory, which is not limited to the interior of the QoS region. The results derived here hold for both power-constrained and unconstrained systems. As will be discussed later, the theory of Yates [8] is a special case of the more general theory proposed here.

Another lesson from the literature is the importance of matrix-based interference models. In virtually any practically relevant system, interference coupling can be characterized with the aforementioned coupling matrix $\mathbf{\Psi}$. Possible receive strategies can be included by assuming a parameter-dependent matrix $\mathbf{\Psi}(z)$, where z stands for the receive strategy. In the beamforming context, discussed in Section 1.2, matrix theory could be applied successfully in order to derive efficient algorithmic solutions.

Thus, another goal of this paper is to generalize the beneficial properties and algorithms observed from the beamforming problem to more general classes of systems, where $\mathbf{\Psi}(z)$ depends on the parameter z in a certain way, as discussed in Section 6.

We will also address a problem that has been neglected in the context of beamforming. Namely, the SIR balancing theory is mostly based on the assumption of nonnegative irreducible matrices. Irreducibility (see, e.g., [22]) is justified in the context of classical power control, when $\mathbf{\Psi}$ consists of strictly positive link gains. However, the impact of the adaptive receive strategy z on $\mathbf{\Psi}(z)$ possibly leads to zero entries. So another contribution of this paper is to analyze the SIR balancing problem for the general case $\mathbf{\Psi}(z) \geq 0$ without the restricting assumption of irreducibility.

Finally, it is desirable to have a unifying theory, which combines the axiomatic framework and the matrix-based theory. Both aspects have been studied separately so far. Viewing concepts from more than one perspective generally produces deeper understanding. In this respect, the results of this paper may prove useful as a basis for the development of future resource allocation concepts.

In this paper, we focus on the SIR balancing aspect, which can be seen as the basis for all interference-related balancing problems. This work will be complemented by [23], where properties of the QoS region are studied, and [24], where we study interference balancing in the presence of noise.

Some notational conventions are matrices and vectors are set in boldface. Let $\mathbf{y}$ be a vector, then $\mathbf{y}_l := [\mathbf{y}]_l$ is the lth component. We use $:=$ for definitions. Finally, $\mathbf{y} \geq 0$ means componentwise inequality, that is, $\mathbf{y}_l \geq 0$ for all indices l.

2. AXIOMATIC INTERFERENCE MODEL

In order to keep the results as general as possible, we do not specify exactly how $\mathcal{I}_k$ depends on $\mathbf{p}$. The mapping can be linear or nonlinear, and it can also model the impact of adaptive receiver designs, like MMSE or interference cancellation. It can also contain noise. The only basic requirement is that the following axioms (A1)–(A3) are fulfilled.

Definition 1. $\mathcal{I}_k : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is called interference function if and only if the following axioms hold:

(A1) $\mathcal{I}_k(\mathbf{p})$ is nonnegative;
(A2) $\mathcal{I}_k(\mu\mathbf{p}) = \mu\mathcal{I}_k(\mathbf{p})$ (scalability);
(A3) $\mathcal{I}_k(\mathbf{p}^{(1)}) \geq \mathcal{I}_k(\mathbf{p}^{(2)})$ if $\mathbf{p}^{(1)} \geq \mathbf{p}^{(2)}$ (monotonicity).

These axioms describe basic properties which are typical for what is usually understood as "interference." Property (A1) follows from the fact that $\mathcal{I}_k$ stands for a power. Property (A2) describes the fact that a scaling of the powers immediately results in a scaling of the received interference. Property (A3) means that by increasing transmission powers, one can never reduce interference.

Many examples are conceivable, like the following examples.

(i) $\mathcal{I}_k(\mathbf{p}) = [\mathbf{\Psi}(z)\mathbf{p}]_k$, where $\mathbf{\Psi}(z)$ is a parameter-dependent nonnegative coupling matrix. This specific model, which holds, for example, for beamforming and CDMA designs, will be discussed later in Section 6.

(ii) $\mathcal{I}_k(\mathbf{p}) = \max_c f_k(\mathbf{p}, c)$, where $f_k(\mathbf{p}, c)$ is the interference for a given power allocation $\mathbf{p}$ under some receiver mismatch c. This definition could be used to model worst-case interference under imperfect channel knowledge.

The continuity of $\mathcal{I}_k(\mathbf{p})$ with respect to $\mathbf{p}$ is an important property, for example, for the development of convergent algorithms. In Section 3, we will show that continuity mostly follows directly from (A2) and (A3). For example, continuity is always fulfilled for $\mathbf{p} > 0$ and some special scenarios discussed later. For all other cases, we require an additional axiom:

(A4) $\mathcal{I}_k(\mathbf{p})$ is continuous on $\mathbb{R}_+^K$.

At first sight, this axiomatic model resembles the concept of standard interference functions introduced by Yates [8]. However, we are interested in asymptotic feasibility, which is only limited by interference. Thus $\mathcal{I}_k(\mathbf{p})$ does not need to contain a noise component as in [8]. The noiseless case is associated with the boundary of the QoS feasible region, whereas the framework in [8] aims at achieving points in the interior of the region. As discussed in the introduction, the framework [8] can also be seen as a special case of the more general approach chosen here. A detailed analysis of interference balancing with noise can be found in [24].

Sometimes, it is necessary to assume that $\mathcal{I}_k(\mathbf{p})$ is strictly positive, for example, in order to ensure that the SIR (2) is defined. Note that this does not restrict the generality of the results. If a user is not affected by interference, then it can be treated separately. Moreover, the following lemma shows that $\mathcal{I}_k(\mathbf{p}) > 0$ need not be required for all $\mathbf{p} > 0$. It is sufficient that there exists *one* positive power allocation such that the interference is strictly positive.

Lemma 1. *If there exists a $\hat{\mathbf{p}} > 0$ such that $\mathcal{I}_k(\hat{\mathbf{p}}) > 0$, then $\mathcal{I}_k(\mathbf{p}) > 0$ for all $\mathbf{p} > 0$.*

Proof. Suppose that $\mathcal{I}_k(\hat{\mathbf{p}}) > 0$. For an arbitrary $\mathbf{p} > 0$, there exists a scalar $\lambda > 0$ such that $\lambda\mathbf{p} > \hat{\mathbf{p}}$. Applying (A2) and (A3), we have $\lambda\mathcal{I}_k(\mathbf{p}) = \mathcal{I}_k(\lambda\mathbf{p}) \geq \mathcal{I}_k(\hat{\mathbf{p}}) > 0$. □

This holds for all kinds of interference functions, even if $\mathcal{I}_k(\mathbf{p})$ includes an adaptive receiver design with interference cancellation or nulling. If the receiver leaves residual interference for one positive power allocation, then it will leave interference for *all* positive power allocations.

Having introduced the QoS model, we are now able to characterize the set of QoSs which are jointly feasible. Consider QoS requirements $Q_1, \dots, Q_K > 0$. Let γ be the inverse function of ϕ, then

$$\gamma_k = \gamma(Q_k), \quad k \in \{1, 2, \dots, K\}, \tag{5}$$

is the minimum SIR level needed by the kth user to satisfy the QoS target Q_k. Thus, the problem of achieving certain QoS requirements carries over to the problem of achieving SIR targets γ_k, which will be summarized by the diagonal matrix

$$\mathbf{\Gamma}_{\mathrm{Q}} = \operatorname{diag}\{\gamma_1, \dots, \gamma_K\}. \tag{6}$$

A target $\mathbf{\Gamma}_{\mathrm{Q}} > 0$ is feasible if and only if there exists a power allocation $\hat{\mathbf{p}} > 0$ such that $\mathrm{SIR}_k(\hat{\mathbf{p}}) \geq \gamma_k$, for all $k = 1, \dots, K$, which is equivalent to $\min_k \mathrm{SIR}_k(\hat{\mathbf{p}})/\gamma_k \geq 1$ or $\max_k \gamma_k/\mathrm{SIR}_k(\hat{\mathbf{p}}) \leq 1$. We have $\max_k \gamma_k/\mathrm{SIR}_k(\hat{\mathbf{p}}) = \max_k \gamma_k\mathcal{I}_k(\mathbf{p})/\mathbf{p}_k$, of which the optimum achievable level is

$$C(\mathbf{\Gamma}_{\mathrm{Q}}) = \inf_{\mathbf{p}>0}\left(\max_{1\leq k\leq K}\frac{\gamma_k\mathcal{I}_k(\mathbf{p})}{\mathbf{p}_k}\right) = \inf_{\substack{\mathbf{p}>0\\ \sum_k \mathbf{p}_k=1}}\left(\max_{1\leq k\leq K}\frac{\gamma_k\mathcal{I}_k(\mathbf{p})}{\mathbf{p}_k}\right). \tag{7}$$

Note that we can restrict the optimization to $\|\mathbf{p}\|_1 = 1$, since $\mathrm{SIR}_k(\mathbf{p})$ is invariant with respect to a scaling of $\mathbf{p}$, as follows from (2) and (A2).

The optimum $C(\mathbf{\Gamma}_{\mathrm{Q}})$ provides a single measure for the joint feasibility of the targets $\mathbf{\Gamma}_{\mathrm{Q}}$, that is, $\mathbf{\Gamma}_{\mathrm{Q}}$ is feasible if and only if $C(\mathbf{\Gamma}_{\mathrm{Q}}) \leq 1$. Thus, the QoS feasible region under the assumption of some (not specified) receive strategy is given as

$$\mathcal{Q} = \{[Q_1, \dots, Q_K] : C(\mathbf{\Gamma}_{\mathrm{Q}}) \leq 1\}. \tag{8}$$

Properties of the boundary of the region characterized by $C(\mathbf{\Gamma}_{\mathrm{Q}}) = 1$ will be studied in the following.

Note that the axiomatic framework based on (A1)–(A4) is very general and includes known results as special cases. In Section 6, we will show that there is an interesting connection to the case where $\mathcal{I}_k(\mathbf{p})$ is composed by a nonnegative coupling matrix. Feasibility and max-min SIR balancing are relatively well understood for this matrix-based model under the assumption that the matrix is strictly positive or irreducible (see, e.g., [4]). The axiomatic model introduced here allows to extend certain properties to a more general class of functions (including the general and less studied case of *reducible* coupling matrices). This not only gives a deeper understanding of the SIR balancing problem, but also provides a generic strategy for handling complex scenarios and cross-layer issues.

3. CONTINUITY

In this section, it will be shown for *arbitrary* axiom-based interference functions $\mathcal{I}_k(\mathbf{p})$ that the requirement (A4) (continuity) often follows as a direct consequence of (A1)–(A3). For all other cases, (A4) is required. Later, in Section 6.1 we will show that continuity *always* holds for a special class of matrix-based interference functions of the form (82).

3.1. General continuity analysis

An important prerequisite for the following results is the continuity of the interference function $\mathcal{I}_k$. Theorem 2 shows that for $\mathbf{p} > 0$, continuity follows directly from the axioms (A2) and (A3). Then, we will show in Theorem 3 how far this can be extended to the case $\mathbf{p} \geq 0$.

Theorem 2. *Consider an arbitrary $\tilde{\mathbf{p}} > 0$ and a sequence $\mathbf{p}^{(n)}$ which converges to $\tilde{\mathbf{p}}$ for $n \to \infty$, then*

$$\lim_{n\to\infty}\mathcal{I}_k(\mathbf{p}^{(n)}) = \mathcal{I}_k(\tilde{\mathbf{p}}), \quad 1 \leq k \leq K. \tag{9}$$

Thus, $\mathcal{I}_k(\mathbf{p})$ is continuous for $\mathbf{p} > 0$.

Proof. Let $\mathbf{p}^{(a)} > 0$, $\mathbf{p}^{(b)} > 0$, and $B = \max_k \mathbf{p}_k^{(a)}/\mathbf{p}_k^{(b)}$, thus, $\mathbf{p}^{(a)} \leq B\mathbf{p}^{(b)}$. Using (A2) and (A3), we have $\mathcal{I}_k(\mathbf{p}^{(a)}) \leq B\mathcal{I}_k(\mathbf{p}^{(b)})$, for all k. This holds for all indices $k = 1, 2, \dots, K$, thus

$$\max_{1\leq k\leq K}\frac{\mathcal{I}_k(\mathbf{p}^{(a)})}{\mathcal{I}_k(\mathbf{p}^{(b)})} \leq B = \max_{1\leq l\leq K}\frac{\mathbf{p}_l^{(a)}}{\mathbf{p}_l^{(b)}}. \tag{10}$$

Similarly, we can define $b = \min_k \mathbf{p}_k^{(a)}/\mathbf{p}_k^{(b)}$ and using (A2) and (A3), we can show that

$$\min_{1\leq k\leq K}\frac{\mathcal{I}_k(\mathbf{p}^{(a)})}{\mathcal{I}_k(\mathbf{p}^{(b)})} \geq b = \min_{1\leq l\leq K}\frac{\mathbf{p}_l^{(a)}}{\mathbf{p}_l^{(b)}}. \tag{11}$$

Now, consider sequences $\mathbf{p}_k^{(n)} \to \tilde{\mathbf{p}}_k$, for all k, where $\tilde{\mathbf{p}} > 0$ is arbitrary. Since $\mathbf{p}^{(n)}$ converges towards $\tilde{\mathbf{p}}$, we can assume that $\mathbf{p}^{(n)} > 0$ without loss of generality. Since $\lim_{n\to\infty}\mathbf{p}_k^{(n)} = \tilde{\mathbf{p}}_k$, we have

$$\begin{aligned}\lim_{n\to\infty}\max_{1\leq k\leq K}\frac{\mathbf{p}_k^{(n)}}{\tilde{\mathbf{p}}_k} &= 1,\\ \lim_{n\to\infty}\min_{1\leq k\leq K}\frac{\mathbf{p}_k^{(n)}}{\tilde{\mathbf{p}}_k} &= 1.\end{aligned} \tag{12}$$

Applying (10) and (11), we have

$$\limsup_{n\to\infty} \max_{1\le k\le K} \frac{\mathcal{I}_k(\mathbf{p}^{(n)})}{\mathcal{I}_k(\widetilde{\mathbf{p}})} \le 1, \qquad \liminf_{n\to\infty} \min_{1\le k\le K} \frac{\mathcal{I}_k(\mathbf{p}^{(n)})}{\mathcal{I}_k(\widetilde{\mathbf{p}})} \ge 1, \tag{13}$$

and thus

$$\lim_{n\to\infty} \frac{\mathcal{I}_k(\mathbf{p}^{(n)})}{\mathcal{I}_k(\widetilde{\mathbf{p}})} = 1, \tag{14}$$

from which (9) follows. □

Theorem 2 shows continuity for $\mathbf{p} > 0$. Next, we study how far this can be generalized to $\mathbf{p} \ge 0$. To this end, consider an arbitrary $\mathbf{p} \ge 0$ and a sequence $\mathbf{p}^{(n)} \ge 0$ with $\lim_{n\to\infty} \mathbf{p}_k^{(n)} = \mathbf{p}_k$, $1 \le k \le K$. Let

$$\bar{\mathbf{p}}_k^{(n)} = \sup_{l\ge n} \mathbf{p}_k^{(l)}, \qquad \underline{\mathbf{p}}_k^{(n)} = \inf_{l\ge n} \mathbf{p}_k^{(l)}. \tag{15}$$

Thus,

$$\underline{\mathbf{p}}_k^{(n)} \le \mathbf{p}_k \le \bar{\mathbf{p}}_k^{(n)}, \quad 1 \le k \le K. \tag{16}$$

From the definition (15), we have $\bar{\mathbf{p}}_k^{(n+1)} \le \bar{\mathbf{p}}_k^{(n)}$, for all k, and $\underline{\mathbf{p}}_k^{(n+1)} \ge \underline{\mathbf{p}}_k^{(n)}$. Thus, for all k, there exist limits

$$\bar{C}_k = \lim_{n\to\infty} \mathcal{I}_k(\bar{\mathbf{p}}^{(n)}), \qquad \underline{C}_k = \lim_{n\to\infty} \mathcal{I}_k(\underline{\mathbf{p}}^{(n)}). \tag{17}$$

Inequality (16) implies that $\bar{C}_k \ge \underline{C}_k$, for all k.

Theorem 3. *Let* $\mathbf{p} \ge 0$ *be fixed and* $\mathbf{p}^{(n)} \ge 0$ *a sequence with* $\lim_{n\to\infty} \mathbf{p}_k^{(n)} = \mathbf{p}_k, 1 \le k \le K$, *then*

$$\lim_{n\to\infty} \mathcal{I}_k(\underline{\mathbf{p}}^{(n)}) = \mathcal{I}_k(\mathbf{p}), \quad \forall k, \tag{18}$$

$$\liminf_{n\to\infty} \mathcal{I}_k(\mathbf{p}^{(n)}) \ge \mathcal{I}_k(\mathbf{p}), \quad \forall k. \tag{19}$$

Proof. First, consider the sequence $\underline{\mathbf{p}}^{(n)}$. If $\mathbf{p}_k = 0$, then $\underline{\mathbf{p}}_k^{(n)} = 0$ for all n. If $\mathbf{p}_k > 0$, then there exists an n_0 such that $\underline{\mathbf{p}}_k^{(n)} > 0$ for all $n \ge n_0$. Because of (16), there exists an n_1 such that for all $n \ge n_1$ and all k with $\mathbf{p}_k > 0$, we always have $\mathbf{p}_k^{(n)} > 0$.

If $\mathcal{I}_k(\mathbf{p}) = 0$, then $\mathcal{I}_k(\underline{\mathbf{p}}^{(n)}) = 0$. This follows from (16) and (A3) and (A1). Thus, (18) has been shown for this special case. It remains to consider the case $\mathcal{I}_k(\mathbf{p}) > 0$. Let $O(\mathbf{p}) = \{k : \mathbf{p}_k > 0\}$. Because of (16), we have

$$\mathcal{I}_k(\mathbf{p}) \ge \mathcal{I}_k(\underline{\mathbf{p}}^{(n)}). \tag{20}$$

Defining

$$\alpha_n = \max_{k\in O(\mathbf{p})} \frac{\mathbf{p}_k}{\underline{\mathbf{p}}_k^{(n)}}, \tag{21}$$

we have

$$\mathcal{I}_k(\mathbf{p}) \le \alpha_n \mathcal{I}_k(\underline{\mathbf{p}}^{(n)}). \tag{22}$$

Since $\lim_{n\to\infty} \underline{\mathbf{p}}_k^{(n)} = \mathbf{p}_k$, for all k, we have $\lim_{n\to\infty} \alpha_n = 1$. Since $\lim_{n\to\infty} \mathcal{I}_k(\underline{\mathbf{p}}^{(n)}) = \underline{C}_k$, we can conclude with (20) and (22) that (18) holds.

Inequality (19) is a consequence of (18) and the fact that $\mathcal{I}_k(\mathbf{p}^{(n)}) \ge \mathcal{I}_k(\underline{\mathbf{p}}^{(n)})$. □

The result (18) in Theorem 3 shows continuity for monotonically increasing sequences. For arbitrary sequences, we only have property (19).

Since $\bar{\mathbf{p}}_k^{(n)} \ge \mathbf{p}_k$, for all k, we have $\mathcal{I}_k(\bar{\mathbf{p}}^{(n)}) \ge \mathcal{I}_k(\mathbf{p})$, for all k. But generally, it is not possible to obtain a lower bound. Although we have $\min_{k\in O(\mathbf{p})} \mathbf{p}_k/\bar{\mathbf{p}}_k^{(n)} = c_n$ and $\lim_{n\to\infty} c_n = 1$, the property $\mathbf{p}_k \ge c_n \bar{\mathbf{p}}_k^{(n)}$, $k \in O(\mathbf{p})$, is not sufficient for finding a lower bound for $\mathcal{I}_k(\mathbf{p})$. Such a bound does not exist for $k \in [1, K]\backslash O(\mathbf{p})$.

3.2. *Continuity for K = 2*

Now, we show that continuity always holds for $K = 2$ under the assumption that no self-interference occurs and that there exists a $\mathbf{p} > 0$ such that $\mathcal{I}_k(\mathbf{p}) > 0$ for all k, which means that the interference functions are guaranteed to be strictly positive (see Lemma 1). Then, the interference $\mathcal{I}_1(\mathbf{p})$ only depends on the power of user 2. This dependency can be expressed by a monotone function $f_1(\mathbf{p}_2) = \mathcal{I}_1(\mathbf{p})$. From (A2), we have $f_1(\lambda\mathbf{p}_2) = \lambda f_1(\mathbf{p}_2)$. That is, the interference scales linearly with the power. The same can be shown for the second user. There exist constants $c_1, c_2 > 0$ such that

$$\mathcal{I}_1(\mathbf{p}) = c_1\mathbf{p}_2, \quad c_1 > 0, \qquad \mathcal{I}_2(\mathbf{p}) = c_2\mathbf{p}_1, \quad c_2 > 0. \tag{23}$$

Thus, the functions $\mathcal{I}_k(\mathbf{p})$ are continuous for $\mathbf{p} \ge 0$.

4. SIR BALANCING THEORY FOR GENERAL INTERFERENCE FUNCTIONS

In this section, we study properties of the interference functions $\mathcal{I}_k(\mathbf{p})$ in their most general form, that is, only the axioms A1–A3 are required (except for a small restriction on self-interference made in Sections 4.3 and 4.4).

Later, in Sections 5 and 6, we will add assumptions on monotonicity behavior and on the structure of $\mathcal{I}_k$, which will allow us to show more specific properties.

4.1. *Comparison of Min-Max and Max-Min optimizations*

Consider the min-max problem (7), which was shown to provide a necessary and sufficient indicator for feasibility. SIR targets $\boldsymbol{\Gamma}_{\mathrm{Q}}$ are feasible if and only if $C(\boldsymbol{\Gamma}_{\mathrm{Q}}) \le 1$.

Sometimes it is useful to consider a modified problem where minimization and maximization are interchanged.

This leads to the max-min formulation

$$c(\mathbf{\Gamma}_{\mathrm{Q}}) = \sup_{\mathbf{p}>0}\left(\min_{1\le k\le K}\frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{\mathbf{p}_k}\right). \tag{24}$$

Note that (7) and (24) need not be equivalent. But if $c(\mathbf{\Gamma}_{\mathrm{Q}}) = C(\mathbf{\Gamma}_{\mathrm{Q}})$ holds, then this often leads to interesting analytical possibilities and insightful interpretations. An example was shown in the context of multiuser beamforming [19], where $c(\mathbf{\Gamma}_{\mathrm{Q}}) = C(\mathbf{\Gamma}_{\mathrm{Q}})$ was ensured by the assumption that interference is modelled by an irreducible coupling matrix. Then, $c(\mathbf{\Gamma}_{\mathrm{Q}}) = C(\mathbf{\Gamma}_{\mathrm{Q}})$ could be used to prove monotonicity and global convergence of an iterative algorithm which converges towards the optimum $C(\mathbf{\Gamma}_{\mathrm{Q}})$.

The following theorem shows the general relation between $C(\mathbf{\Gamma}_{\mathrm{Q}})$ and $c(\mathbf{\Gamma}_{\mathrm{Q}})$. Later, in Section 6.3 we will discuss a specific scenario for which equality holds.

Theorem 4. *The min-max optimum $C(\mathbf{\Gamma}_{\mathrm{Q}})$ is an upper bound of the max-min optimum $c(\mathbf{\Gamma}_{\mathrm{Q}})$, that is,*

$$c(\mathbf{\Gamma}_{\mathrm{Q}}) \le C(\mathbf{\Gamma}_{\mathrm{Q}}). \tag{25}$$

Proof. Because of the definition (7), there exists a $\bar{\mathbf{p}}^{(\epsilon)} > 0$, for every $\epsilon > 0$, such that

$$\frac{\gamma_k \mathcal{I}_k(\bar{\mathbf{p}}^{(\epsilon)})}{\bar{\mathbf{p}}_k^{(\epsilon)}} \le C(\mathbf{\Gamma}_{\mathrm{Q}}) + \epsilon \quad \forall k \in \{1,2,\dots,K\}. \tag{26}$$

Definition (24) implies the existence of a $\underline{\mathbf{p}}^{(\epsilon)} > 0$, for every $\epsilon > 0$, such that

$$\frac{\gamma_k \mathcal{I}_k(\underline{\mathbf{p}}^{(\epsilon)})}{\underline{\mathbf{p}}_k^{(\epsilon)}} \ge c(\mathbf{\Gamma}_{\mathrm{Q}}) - \epsilon \quad \forall k \in \{1,2,\dots,K\}. \tag{27}$$

Since $\mathrm{SIR}_k(\mathbf{p})$ is invariant to a scaling of $\mathbf{p}$, we can assume that

$$\bar{\mathbf{p}}_k^{(\epsilon)} \ge \underline{\mathbf{p}}_k^{(\epsilon)} \quad \forall k \in \{1,2,\dots,K\}, \tag{28}$$

and there exists an index k_0 such that $\bar{\mathbf{p}}_{k_0}^{(\epsilon)} = \underline{\mathbf{p}}_{k_0}^{(\epsilon)}$. Thus,

$$\begin{aligned} C(\mathbf{\Gamma}_{\mathrm{Q}}) + \epsilon &\ge \max_{1\le k\le K}\frac{\gamma_k \mathcal{I}_k(\bar{\mathbf{p}}^{(\epsilon)})}{\bar{\mathbf{p}}_k^{(\epsilon)}} \\ &\ge \frac{\gamma_{k_0} \mathcal{I}_{k_0}(\bar{\mathbf{p}}^{(\epsilon)})}{\bar{\mathbf{p}}_{k_0}^{(\epsilon)}} = \frac{\gamma_{k_0} \mathcal{I}_{k_0}(\bar{\mathbf{p}}^{(\epsilon)})}{\underline{\mathbf{p}}_{k_0}^{(\epsilon)}}. \end{aligned} \tag{29}$$

From (28) and (A3), we know that $\mathcal{I}_k(\bar{\mathbf{p}}^{(\epsilon)}) \ge \mathcal{I}_k(\underline{\mathbf{p}}^{(\epsilon)})$, for all k, and thus

$$C(\mathbf{\Gamma}_{\mathrm{Q}}) + \epsilon \ge \frac{\gamma_{k_0} \mathcal{I}_{k_0}(\underline{\mathbf{p}}^{(\epsilon)})}{\underline{\mathbf{p}}_{k_0}^{(\epsilon)}} \ge \min_{1\le k\le K}\frac{\gamma_k \mathcal{I}_k(\underline{\mathbf{p}}^{(\epsilon)})}{\underline{\mathbf{p}}_k^{(\epsilon)}} \ge c(\mathbf{\Gamma}_{\mathrm{Q}}) - \epsilon, \tag{30}$$

which concludes the proof. □

4.2. *Achievability of SIR targets*

Next, we study power allocations which are optimal with respect to the min-max optimization goal (7). Namely, we are interested in vectors $\mathbf{p} > 0$, which minimize $\max_k \gamma_k/\mathrm{SIR}_k(\mathbf{p})$. We assume that there exists a $\mathbf{p} > 0$ such that $\mathcal{I}_k(\mathbf{p}) > 0$, for all k, so $\mathrm{SIR}_k(\mathbf{p})$ is always defined (see Lemma 1).

Note that the existence of an optimizer, $\mathbf{p} > 0$, is not always guaranteed. It can happen that the infimum $C(\mathbf{\Gamma}_{\mathrm{Q}})$, as in (7), is not achieved, but approached by $\max_k \gamma_k/\mathrm{SIR}_k(\mathbf{p})$ arbitrarily close, so all quantities γ_k/SIR_k are asymptotically balanced at the common level $C(\mathbf{\Gamma}_{\mathrm{Q}})$. In this sense, the expression "SIR balancing" is justified.

An alternative way of expressing this balanced state is to use the fixed-point equation $\gamma_k \mathcal{I}_k(\mathbf{p}) = \mu \mathbf{p}_k$, for all k. This has the advantage that zeros in the power allocations can be admitted. The existence of power allocations $\mathbf{p} \ge 0$ (excluding the trivial all-zero allocation $\mathbf{p} = \mathbf{0}$) will be characterized in the following by Lemma 5 and Theorem 6, which show that components of the power allocation $\mathbf{p}$ can tend to zero, in which case also the interference tends to zero (because of the min-max principle).

We start by considering the function

$$\mathcal{I}_k(\mathbf{p},\epsilon) = \mathcal{I}_k(\mathbf{p}) + \epsilon\sum_k \mathbf{p}_k, \quad 1 \le k \le K, \tag{31}$$

with $\epsilon > 0$. The resulting min-max optimum is

$$C(\mathbf{\Gamma}_{\mathrm{Q}},\epsilon) = \inf_{\mathbf{p}>0}\left(\max_{1\le k\le K}\frac{\gamma_k \mathcal{I}_k(\mathbf{p},\epsilon)}{\mathbf{p}_k}\right). \tag{32}$$

The following lemma will be needed in the following.

Lemma 5. *For every $\epsilon > 0$, there exists a $\mathbf{p}^{(\epsilon)} > 0$ such that*

$$\gamma_k \mathcal{I}_k(\mathbf{p}^{(\epsilon)},\epsilon) = C(\mathbf{\Gamma}_{\mathrm{Q}},\epsilon)\mathbf{p}_k^{(\epsilon)}, \quad 1 \le k \le K. \tag{33}$$

Proof. From (32), we know that for every $\delta > 0$, there exists a vector $\hat{\mathbf{p}}^{(\delta)} > 0$ with $\|\hat{\mathbf{p}}^{(\delta)}\|_1 = 1$, such that

$$\max_{1\le k\le K}\frac{\gamma_k \mathcal{I}_k(\hat{\mathbf{p}}^{(\delta)},\epsilon)}{\hat{\mathbf{p}}_k^{(\delta)}} \le C(\mathbf{\Gamma}_{\mathrm{Q}},\epsilon) + \delta. \tag{34}$$

The inequality holds for all $k \in \{1,2,\dots,K\}$. Thus, using definition (31), we have

$$\gamma_k\epsilon \le \gamma_k \mathcal{I}_k(\hat{\mathbf{p}}^{(\delta)}) + \gamma_k\epsilon\sum_k \hat{\mathbf{p}}_k^{(\delta)} \le \left(C(\mathbf{\Gamma}_{\mathrm{Q}},\epsilon) + \delta\right)\hat{\mathbf{p}}_k^{(\delta)}. \tag{35}$$

There exists a sequence $\{\delta_n\}$, $\delta_n \to 0$, and $\hat{\mathbf{p}} \ge 0$, such that

$$\lim_{n\to\infty}\sum_{k=1}^{K}\left|\hat{\mathbf{p}}_k^{(\delta_n)} - \hat{\mathbf{p}}_k\right| = 0. \tag{36}$$

Inequality (35) implies that for all $1 \le k \le K$,

$$\begin{aligned} \lim_{n\to\infty}\gamma_k \mathcal{I}_k(\hat{\mathbf{p}}^{(\delta_n)},\epsilon) &\le \lim_{n\to\infty}\left(C(\mathbf{\Gamma}_{\mathrm{Q}},\epsilon) + \delta_n\right)\hat{\mathbf{p}}_k^{(\delta_n)} \\ &= C(\mathbf{\Gamma}_{\mathrm{Q}},\epsilon)\hat{\mathbf{p}}_k, \quad 1 \le k \le K. \end{aligned} \tag{37}$$

Combining (35) and (37), we have

$$0 < \gamma_k \epsilon \le C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon)\hat{\mathbf{p}}_k, \quad 1 \le k \le K. \tag{38}$$

Thus, $\hat{\mathbf{p}} > 0$. Note that $\hat{\mathbf{p}}$ depends on ϵ.

It remains to show the equality (33). From (37), we know that

$$\gamma_k \mathcal{I}_k(\hat{\mathbf{p}}, \epsilon) \le C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon)\hat{\mathbf{p}}_k, \quad 1 \le k \le K. \tag{39}$$

Now, suppose that there exists a k_0 such that

$$\gamma_{k_0} \mathcal{I}_{k_0}(\hat{\mathbf{p}}, \epsilon) < C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon)\hat{\mathbf{p}}_{k_0}. \tag{40}$$

Then it would be possible to reduce the value $\hat{\mathbf{p}}_{k_0}$ and to reduce all other links $k \neq k_0$, that is,

$$C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon) > \inf_{\mathbf{p} > 0} \max_{1 \le k \le K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p}, \epsilon)}{\mathbf{p}_k}, \tag{41}$$

which is a contradiction. □

Lemma 5 shows that a balanced optimum can always be achieved with the modified interference functions $\mathcal{I}_k(\mathbf{p}, \epsilon)$ and $\mathbf{p} > 0$. Letting $\epsilon \to 0$, we can show the following result.

Theorem 6. *There always exists a vector $\mathbf{p}^* \ge 0$, $\mathbf{p}^* \neq \mathbf{0}$, such that*

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*) = C(\mathbf{\Gamma}_{\mathrm{Q}})\mathbf{p}_k^*, \quad 1 \le k \le K. \tag{42}$$

Proof. For $0 < \epsilon_1 < \epsilon_2$, we have $\mathcal{I}_k(\mathbf{p}, \epsilon_1) < \mathcal{I}_k(\mathbf{p}, \epsilon_2)$, and thus

$$C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon_1) \le C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon_2). \tag{43}$$

Since $C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon)$ is nonnegative, the limit $M = \lim_{\epsilon \to 0} C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon)$ exists.

First, we show that $M = C(\mathbf{\Gamma}_{\mathrm{Q}})$. Since $\mathcal{I}_k(\mathbf{p}) \le \mathcal{I}_k(\mathbf{p}, \epsilon)$, $1 \le k \le K$, we have

$$M \ge C(\mathbf{\Gamma}_{\mathrm{Q}}). \tag{44}$$

It is known from (7) that for every $\delta > 0$, there exists a vector $\mathbf{p}^{(\delta)} > 0$, with $\|\mathbf{p}^{(\delta)}\|_1 = 1$, such that

$$\max_{1 \le k \le K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p}^{(\delta)})}{\mathbf{p}_k^{(\delta)}} \le C(\mathbf{\Gamma}_{\mathrm{Q}}) + \delta. \tag{45}$$

The inequality is fulfilled for all indices $1 \le k \le K$, thus

$$\begin{aligned} \frac{\gamma_k \mathcal{I}_k(\mathbf{p}^{(\delta)}, \epsilon)}{\mathbf{p}_k^{(\delta)}} &= \frac{\gamma_k \mathcal{I}_k(\mathbf{p}^{(\delta)})}{\mathbf{p}_k^{(\delta)}} + \frac{\gamma_k \epsilon}{\mathbf{p}_k^{(\delta)}} \\ &\le C(\mathbf{\Gamma}_{\mathrm{Q}}) + \delta + \frac{\gamma_k \epsilon}{\mathbf{p}_k^{(\delta)}}, \quad 1 \le k \le K. \end{aligned} \tag{46}$$

It follows that

$$\begin{aligned} M \le C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon) &\le \max_k \frac{\gamma_k \mathcal{I}_k(\mathbf{p}^{(\delta)}, \epsilon)}{\mathbf{p}_k^{(\delta)}} \\ &\le C(\mathbf{\Gamma}_{\mathrm{Q}}) + \delta + \epsilon \max_{1 \le k \le K} \frac{\gamma_k}{\mathbf{p}_k^{(\delta)}}. \end{aligned} \tag{47}$$

For $\epsilon \to 0$, we have $M \le C(\mathbf{\Gamma}_{\mathrm{Q}}) + \delta$, which holds for all $\delta > 0$. Thus, $M \le C(\mathbf{\Gamma}_{\mathrm{Q}})$, which implies that the inequality (44) must be fulfilled with equality, that is,

$$\lim_{\epsilon \to 0} C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon) = C(\mathbf{\Gamma}_{\mathrm{Q}}). \tag{48}$$

We know from Lemma 5 that for every $\epsilon > 0$, there exists a $\mathbf{p}^*(\epsilon) > 0$ such that

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*(\epsilon), \epsilon) = C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon)\mathbf{p}_k^*(\epsilon), \quad 1 \le k \le K. \tag{49}$$

Since $\|\mathbf{p}^*(\epsilon)\|_1 = 1$ can be assumed, and $\mathbf{p}^*(\epsilon) > 0$, there exists a subsequence $\{\epsilon_n\}$ and a $\mathbf{p}^* \ge 0$ such that

$$\lim_{n \to \infty} \sum_{k=1}^{K} |\mathbf{p}_k^*(\epsilon_n) - \mathbf{p}_k^*| = 0. \tag{50}$$

With (48), the continuity of $\mathcal{I}_k$ in Theorem 2, and (49), we have

$$\begin{aligned} C(\mathbf{\Gamma}_{\mathrm{Q}})\mathbf{p}_k^* &= \lim_{n \to \infty} C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon_n)\mathbf{p}_k^*(\epsilon_n) \\ &= \lim_{n \to \infty} \gamma_k \mathcal{I}_k(\mathbf{p}^*(\epsilon_n), \epsilon_n) \\ &= \lim_{n \to \infty} \gamma_k \mathcal{I}_k(\mathbf{p}(\epsilon_n)) \\ &= \gamma_k \mathcal{I}_k(\mathbf{p}^*), \quad 1 \le k \le K, \end{aligned} \tag{51}$$

which concludes the proof. □

Theorem 6 shows that there always exists an allocation $\mathbf{p} \ge 0$ with nonzero power components such that the ratios $\mathrm{SIR}_k(\mathbf{p})/\gamma_k$ are balanced at the same level. Later, in Theorem 11 it will be shown that if there exists a $\mathbf{p}^* > 0$ such that (42) is fulfilled, then $\mathbf{p}^*$ is the optimizer of the SIR balancing problem (7). Otherwise, the infimum (7) is only approached asymptotically and no optimizer exists. Then, the quantities $\mathrm{SIR}_k(\mathbf{p})/\gamma_k$ are only balanced asymptotically. In this case, we know from Theorem 6 that the balanced state can be characterized by allowing power components equal to zero.

4.3. Additional properties of the solution

In this section, we show additional properties of the optimizers under the assumption that

(1) no self-interference occurs;
(2) for each index k, there exists a $\mathbf{p} > 0$ such that $\mathcal{I}_k(\mathbf{p}) > 0$.

In this case, we know from Lemma 1 that for all $\mathbf{p} > 0$, the interference functions are strictly positive.

Theorem 7. *Suppose that (1) and (2) are fulfilled. Also, $\mathbf{p}' \ge 0$ fulfills $\gamma_k \mathcal{I}_k(\mathbf{p}') = C(\mathbf{\Gamma}_{\mathrm{Q}})\mathbf{p}'_k$, for all k, and there exists an index k_0 such that $\mathbf{p}'_{k_0} = 0$, then $\mathbf{p}'$ has at least two zero components.*

Proof. Suppose that $\mathbf{p}'_{k_0} = 0$ is the only zero component. Because of (1) and (2), we have $\mathcal{I}_{k_0}(\mathbf{p}') > 0$, which leads to the contradiction $0 < \gamma_{k_0} \mathcal{I}_{k_0}(\mathbf{p}') = C(\mathbf{\Gamma}_{\mathrm{Q}})\mathbf{p}'_{k_0} = 0$. □

Theorem 8. *Suppose that (1) and (2) are fulfilled. For $K = 2, 3$, there exists exactly one vector $\mathbf{p} \geq 0$, $\mathbf{p} \neq \mathbf{0}$, such that $C(\mathbf{\Gamma}_Q)\mathbf{p}_k = \gamma_k \mathcal{I}_k(\mathbf{p})$, for all k, and this vector fulfills $\mathbf{p} > 0$.*

Proof. From Theorem 6, we know that there always exists a $\mathbf{p} \geq 0$ such that (42) is fulfilled. If there exists a k_0 such that $\mathbf{p}_{k_0} = 0$, then it follows from Theorem 7 that the vector $\mathbf{p}$ has at least two zero entries.

For $K = 2$, we know that $\mathcal{I}_1(\mathbf{p})$ and $\mathcal{I}_2(\mathbf{p})$ are reduced to (23), respectively. Thus, there exists exactly one vector $\mathbf{p} \geq 0$ such that $C(\mathbf{\Gamma}_Q)\mathbf{p}_k = \gamma_k \mathcal{I}_k(\mathbf{p})$, $k = 1, 2$, and this vector is strictly positive, that is, $\mathbf{p} > 0$.

For $K = 3$, each vector $\mathbf{p}$ satisfying $C(\mathbf{\Gamma}_Q)\mathbf{p}_k = \gamma_k \mathcal{I}_k(\mathbf{p})$, for all k, is strictly positive, that is, $\mathbf{p} > 0$. The reason is Theorem 7, which shows that only exactly two components can be zero (excluding the trivial all-zero vector). Without loss of generality, assume that $\mathbf{p} = [0, 0, \mathbf{p}_3]$, $\mathbf{p}_3 > 0$. Because of (1), we have

$$C(\mathbf{\Gamma}_Q)\mathbf{p}_3 = \gamma_3 \mathcal{I}_3(\mathbf{p}) = \gamma_3 \mathcal{I}_3([0, 0, \mathbf{p}_3]) = 0, \tag{52}$$

which leads to the contradiction $\mathbf{p}_3 = 0$. Thus, all components are strictly positive.

It remains to show uniqueness. The proof is by contradiction. Suppose that there exist $\mathbf{p}^{(1)}, \mathbf{p}^{(2)} > 0$. Without loss of generality, we can assume that $\mathbf{p}^{(1)} \leq \mathbf{p}^{(2)}$ and $\mathbf{p}_1^{(1)} = \mathbf{p}_1^{(2)}$. If $\mathbf{p}_2^{(1)} < \mathbf{p}_2^{(2)}$ and $\mathbf{p}_3^{(1)} < \mathbf{p}_3^{(2)}$, then there exists a $\lambda > 1$ such that $\lambda \mathbf{p}_2^{(1)} < \mathbf{p}_2^{(2)}$ and $\lambda \mathbf{p}_3^{(1)} < \mathbf{p}_3^{(2)}$, and thus

$$\mathcal{I}_1(\lambda \mathbf{p}^{(1)}) = \lambda \mathcal{I}_1(\mathbf{p}_2^{(1)}, \mathbf{p}_3^{(1)}) \leq \mathcal{I}_1(\mathbf{p}^{(2)}). \tag{53}$$

We can conclude that $\mathcal{I}_1(\mathbf{p}^{(1)}) < \mathcal{I}_1(\mathbf{p}^{(2)})$, and thus

$$C(\mathbf{\Gamma}_Q) = \frac{\gamma_1 \mathcal{I}_1(\mathbf{p}^{(1)})}{\mathbf{p}_1^{(1)}} = \frac{\gamma_1 \mathcal{I}_1(\mathbf{p}^{(1)})}{\mathbf{p}_1^{(2)}} < \frac{\gamma_1 \mathcal{I}_1(\mathbf{p}^{(2)})}{\mathbf{p}_1^{(2)}} = C(\mathbf{\Gamma}_Q) \tag{54}$$

which contradicts the existence of two different components.

It remains to contradict the existence of one different component. Without loss of generality, assume that $\mathbf{p}_3^{(1)} < \mathbf{p}_3^{(2)}$, while the first two components are equal. This implies that $\mathcal{I}_3(\mathbf{p}^{(1)}) = \mathcal{I}_3(\mathbf{p}^{(2)})$, and thus

$$C(\mathbf{\Gamma}_Q) = \frac{\gamma_3 \mathcal{I}_3(\mathbf{p}^{(1)})}{\mathbf{p}_3^{(1)}} = \frac{\gamma_3 \mathcal{I}_3(\mathbf{p}^{(2)})}{\mathbf{p}_3^{(1)}} > \frac{\gamma_3 \mathcal{I}_3(\mathbf{p}^{(2)})}{\mathbf{p}_3^{(2)}} = C(\mathbf{\Gamma}_Q) \tag{55}$$

which is a contradiction and shows that $\mathbf{p}^{(1)} = \mathbf{p}^{(2)}$ for all components. □

The boundary of the SIR feasible region is characterized by $C(\mathbf{\Gamma}_Q) = 1$. For $K = 2, 3$, it follows from the above results that all boundary points are always effectively achievable, that is, there always exists a $\mathbf{p} > 0$ such that $\gamma_k \mathcal{I}_k(\mathbf{p}) = \mathbf{p}_k$, for all k. This need not be true for $K \geq 4$, as shown by the following example.

Consider the function $\mathcal{I}_k(\mathbf{p}) = [\boldsymbol{B}\mathbf{p}]_k$, where

$$\boldsymbol{B} = \begin{bmatrix} 0 & b & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \end{bmatrix}. \tag{56}$$

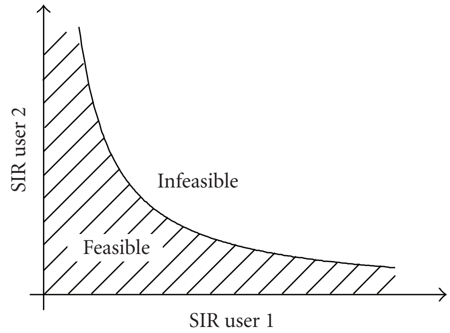

Figure 1: The infeasible SIR region is convex for $K = 2$.

We choose the target $\mathbf{\Gamma}_Q$ such that $C(\mathbf{\Gamma}_Q) = 1$, that is, $\gamma_k = 1/b$, for all k. Then, $\gamma_k \mathcal{I}_k(\mathbf{p}) = \mathbf{p}_k$, for all k, is fulfilled, for example, by the vectors [1111] or [0011]. Thus, there exist different power allocations, which can be strictly positive or not. From Theorem 7, we know that such a behavior can only occur for $K \geq 4$.

The following theorem is interesting in the context of strict positivity. It shows that ambiguities in the power allocation can only exist under certain conditions.

Theorem 9. *Suppose that (1) and (2) are fulfilled. Let K be arbitrary. Suppose that there are two vectors $\mathbf{p}^{(1)}, \mathbf{p}^{(2)} > 0$ such that $\gamma_k \mathcal{I}_k(\mathbf{p}) = C(\mathbf{\Gamma}_Q)\mathbf{p}_k$, for all k. Without loss of generality, $\mathbf{p}^{(1)} \geq \mathbf{p}^{(2)}$ can be ensured by scaling, where equality holds for one component. Then equality holds for at least two components.*

Proof. The proof is in analogy to the proof of Theorem 8 for $K = 3$. □

4.4. Geometrical interpretation

For $K = 2$, the results allow for an interesting geometrical interpretation. Using (23), we have

$$C(\mathbf{\Gamma}_Q)\mathbf{p} = \mathbf{\Gamma}_Q \begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix} \mathbf{p}, \quad c_1, c_2 > 0. \tag{57}$$

The boundary of the feasible region is the set of $\mathbf{\Gamma}_Q = \operatorname{diag}\{\gamma_1, \gamma_2\}$ for which $C(\mathbf{\Gamma}_Q) = 1$. Thus, the boundary is described by

$$\gamma_2 = \frac{1}{c_1 c_2 \gamma_1}. \tag{58}$$

It follows that the infeasible SIR region for $K = 2$ is convex (see Figure 1).

This was already observed in the context of power control [4, 7] and multiuser beamforming [25]. Here we show that this result extents to more general classes of receiver designs. However, this property need not hold for $K \geq 4$, as was recently shown in [7].

4.5. Achievable balanced SIR margin

Theorem 6 shows that the SIR balancing problem (7) leads (at least asymptotically) to a solution $\mathbf{p} \geq 0$ characterized by (42).

In this section, we investigate the nonlinear equation (42) and how it is related to the optimum (7). Note that $\mathbf{p}_k = 0$ means that the kth user is switched off, thus no interference is caused by this user. In general, this means that better SIR levels might be achievable for the other users.

Theorem 10. *Let $\mu > 0$ and $\mathbf{p}^* \geq 0$ fulfill*

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*) = \mu \mathbf{p}_k^*, \quad 1 \leq k \leq K, \tag{59}$$

then $\mu \leq C(\mathbf{\Gamma}_Q)$.

Proof. The result is shown by contradiction. Suppose that $\mu > C(\mathbf{\Gamma}_Q)$, then the definition (7) implies the existence of a vector $\bar{\mathbf{p}} > 0$ such that

$$\gamma_k \mathcal{I}_k(\bar{\mathbf{p}}) < \mu \bar{\mathbf{p}}_k, \quad 1 \leq k \leq K. \tag{60}$$

This relation holds for all vectors $c\bar{\mathbf{p}}$ with $c > 0$. Now, we can choose c such that $c\bar{\mathbf{p}}_k \geq \mathbf{p}_k^*$, for all k, where $\mathbf{p}^* > 0$ fulfills (59), and $c\bar{\mathbf{p}}_{k_0} = \mathbf{p}_{k_0}^*$ for one arbitrary component k_0. Defining $\tilde{\mathbf{p}} := c\bar{\mathbf{p}}$, we have

$$\mu = \frac{\gamma_{k_0} \mathcal{I}_{k_0}(\mathbf{p}^*)}{\mathbf{p}_{k_0}^*} = \frac{\gamma_{k_0} \mathcal{I}_{k_0}(\mathbf{p}^*)}{\tilde{\mathbf{p}}_{k_0}} \leq \frac{\gamma_{k_0} \mathcal{I}_{k_0}(\tilde{\mathbf{p}})}{\tilde{\mathbf{p}}_{k_0}} < \mu, \tag{61}$$

which is a contradiction and concludes the proof. □

The following example shows that the theorem is strict in a sense that it cannot be improved even for the simple case where $\mathcal{I}_k(\mathbf{p})$ is based on a matrix. In particular, the case $\mu < C(\mathbf{\Gamma}_Q)$ is possible.

To this end, consider the function $\mathcal{I}_k(\mathbf{p}) = [\mathbf{\Psi}\mathbf{p}]_k$, where $\mathbf{\Psi} = \left[\begin{smallmatrix} \mathbf{\Psi}^{(1)} & 0 \\ \hat{\mathbf{\Psi}} & \mathbf{\Psi}^{(2)} \end{smallmatrix}\right]$, with $\mathbf{\Psi}^{(1)} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\mathbf{\Psi}^{(2)} = \left[\begin{smallmatrix} 0 & \mu \\ \mu & 0 \end{smallmatrix}\right]$, $0 < \mu < 1$. Then, there exists an eigenvector $\tilde{\mathbf{p}} \geq 0$ such that

$$\mathbf{\Psi}\tilde{\mathbf{p}} = \mu\tilde{\mathbf{p}}, \quad \tilde{\mathbf{p}} = [0, 0, 1, 1]^T. \tag{62}$$

But there also exists a strictly positive eigenvector $\hat{\mathbf{p}} > 0$ such that

$$\mathbf{\Psi}\hat{\mathbf{p}} = \hat{\mathbf{p}}, \quad \hat{\mathbf{p}} = [1, 1, a, b]^T, \tag{63}$$

where a and b solve the equations

$$\begin{aligned} a &= \mu b + \hat{\mathbf{\Psi}}_{11} + \hat{\mathbf{\Psi}}_{12}, \\ b &= \mu a + \hat{\mathbf{\Psi}}_{21} + \hat{\mathbf{\Psi}}_{22}. \end{aligned} \tag{64}$$

This example shows that there can exist different allocations $\mathbf{p} \geq 0$ such that (59) is fulfilled. In particular, it is possible to achieve a level $\mu < C(\mathbf{\Gamma}_Q)$. However, this requires that users are switched off (zero power).

Now, Theorem 11 shows that if there exists a $\mathbf{p} > 0$ which balances all SIR, then $\mu = C(\mathbf{\Gamma}_Q)$.

Theorem 11. *Suppose that there exist a $\mu > 0$ and $\mathbf{p}^* > 0$ such that*

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*) = \mu \mathbf{p}_k^*, \quad 1 \leq k \leq K, \tag{65}$$

then $\mu = C(\mathbf{\Gamma}_Q)$.

Proof. In Theorem 10, it was shown that $\mu \leq C(\mathbf{\Gamma}_Q)$, thus it remains to show equality.

We know from Theorem (42) that there exists a vector $\hat{\mathbf{p}} \geq \mathbf{0}$, $\hat{\mathbf{p}} \neq \mathbf{0}$, such that

$$\gamma_k \mathcal{I}_k(\hat{\mathbf{p}}) = C(\mathbf{\Gamma}_Q)\hat{\mathbf{p}}_k, \quad 1 \leq k \leq K. \tag{66}$$

Each scaled version of $\mathbf{p}^*$ fulfills (65), thus we can choose $\mathbf{p}_k^* \geq \hat{\mathbf{p}}_k$, for all k, and $\mathbf{p}_{k_0}^* = \hat{\mathbf{p}}_{k_0} > 0$ for some index k_0. Thus,

$$C(\mathbf{\Gamma}_Q) = \frac{\gamma_{k_0} \mathcal{I}_{k_0}(\hat{\mathbf{p}})}{\hat{\mathbf{p}}_{k_0}} = \frac{\gamma_{k_0} \mathcal{I}_{k_0}(\hat{\mathbf{p}})}{\mathbf{p}_{k_0}^*} \leq \frac{\gamma_{k_0} \mathcal{I}_{k_0}(\mathbf{p}^*)}{\mathbf{p}_{k_0}^*} = \mu. \tag{67}$$

Thus $\mu \leq C(\mathbf{\Gamma}_Q)$ can only be fulfilled with equality. □

It can be concluded that the SIR balancing problem (7) is equivalent to the problem of finding the maximum μ such that $\gamma_k \mathcal{I}_k(\mathbf{p}) = \mu \cdot \mathbf{p}_k$, $\mathbf{p} > 0$.

Assume that $C^{(1)}(\mathbf{\Gamma}_Q)$ is the balanced optimum (7) for interference functions $\mathcal{I}_k^{(1)}(\mathbf{p})$, and $C^{(2)}(\mathbf{\Gamma}_Q)$ is the optimum for interference functions $\mathcal{I}_k^{(2)}(\mathbf{p})$. If $\mathcal{I}_k^{(1)}(\mathbf{p}) \geq \mathcal{I}_k^{(2)}(\mathbf{p})$ for $\mathbf{p} > 0$, then $C^{(1)}(\mathbf{\Gamma}_Q) \geq C^{(2)}(\mathbf{\Gamma}_Q)$. This is clear from the min-max characterization (7). An interesting observation is that this property immediately transfers to functions $\mathcal{I}_k(\mathbf{p}) = [\mathbf{\Psi}\mathbf{p}]_k$, where $\mathbf{\Psi}$ is a nonnegative coupling matrix. In this case, the optimum $C(\mathbf{\Gamma}_Q)$ can be interpreted as the spectral radius of a coupling matrix $\mathbf{\Gamma}_Q\mathbf{\Psi}$. Thus, element-wise monotonicity $\mathbf{\Psi}_{kl}^{(1)} \geq \mathbf{\Psi}_{kl}^{(2)}$ implies that $\rho(\mathbf{\Gamma}_Q\mathbf{\Psi}^{(1)}) \geq \rho(\mathbf{\Gamma}_Q\mathbf{\Psi}^{(2)})$. This result, which is a byproduct of the max-min approach, would otherwise be more difficult to prove.

4.6. Generalized achievability of SIR targets

So far, we have focused on the existence of power allocations $\mathbf{p}$ which fulfill the equations $\gamma_k \mathcal{I}_k(\mathbf{p}) = C(\mathbf{\Gamma}_Q)\mathbf{p}_k$, for all k. Without loss of generality, we can assume that $\mathbf{\Gamma}_Q$ is a boundary point, that is, $C(\mathbf{\Gamma}_Q) = 1$. Thus, if the equations are fulfilled by $\mathbf{p}^* > 0$, then $\text{SIR}_k(\mathbf{p}^*) = \gamma_k$, for all k. The following set $\mathcal{P}_E(\mathbf{\Gamma}_Q)$ contains all power allocations which achieve $\mathbf{\Gamma}_Q$ with equality:

$$\mathcal{P}_E(\mathbf{\Gamma}_Q) = \{\mathbf{p} > 0 : \gamma_k \mathcal{I}_k(\mathbf{p}) = \mathbf{p}_k \,\forall k\}. \tag{68}$$

From a practical point of view, it is not necessary to require equality. The actual SIR can be larger than the target, that is, $\text{SIR}_k(\mathbf{p}) > \gamma_k$. This seems to be a waste of resources since the target is overfulfilled. However, there are cases where $\text{SIR}_k(\mathbf{p}) \geq \gamma_k$ cannot be fulfilled with equality (see the example at the end of this section). This is a peculiarity of the noiseless case, where $\text{SIR}_k(\mathbf{p})$ is not affected by a scaling of $\mathbf{p}$.

Thus, we will also consider the set $\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}})$, which contains all positive power allocations for which $\text{SIR}_k(\mathbf{p}) \geq \gamma_k$:

$$\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}}) = \{\mathbf{p} > 0 : \gamma_k \mathcal{I}_k(\mathbf{p}) \leq \mathbf{p}_k, \forall k\}. \tag{69}$$

We have $\mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}}) \subseteq \mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}})$. Both sets can be empty.

In the following, we will use a general approach to characterize $\mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}})$, which is based on the behavior of iterations of the interference function. To this end, consider the vector-valued mapping

$$\mathcal{V}(\mathbf{p}) = \begin{bmatrix} \gamma_1 \mathcal{I}_1(\mathbf{p}) \\ \vdots \\ \gamma_K \mathcal{I}_K(\mathbf{p}) \end{bmatrix} \tag{70}$$

and the set

$$\mathcal{V}(\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}})) = \{\mathbf{p} > 0 : \exists \widetilde{\mathbf{p}} \in \mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}}) \text{ with } \mathbf{p} = \mathcal{V}(\widetilde{\mathbf{p}})\}. \tag{71}$$

Each $\widetilde{\mathbf{p}} \in \mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}})$ fulfills $\widetilde{\mathbf{p}} > 0$. We assume that the interference functions fulfill the property stated in Lemma 1, thus we have strictly positive interference functions $\mathcal{I}_k(\widetilde{\mathbf{p}}) > 0$, for all k. Moreover, $\widetilde{\mathbf{p}} \geq \mathcal{V}(\widetilde{\mathbf{p}})$ follows from the definition (69). Thus, applying $\mathcal{V}$ recursively to $\widetilde{\mathbf{p}}$ leads to a monotonically decreasing sequence $\widetilde{\mathbf{p}} \geq \mathcal{V}(\widetilde{\mathbf{p}}) \geq \mathcal{V}(\mathcal{V}(\widetilde{\mathbf{p}})) \geq \cdots$. Applying the mapping l times to the set $\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}})$, we have

$$\mathcal{V}^l(\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}})) \subseteq \mathcal{V}^{l-1}(\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}})). \tag{72}$$

Theorem 12. *$\mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}}) \neq \emptyset$ if and only if $\bigcap_{l=1}^{\infty} \mathcal{V}^l(\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}}) \neq \emptyset$.*

Proof. Define $\underline{\mathcal{P}_O} = \bigcap_{l=1}^{\infty} \mathcal{V}^l(\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}}))$. We have $\mathcal{V}(\mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}})) = \mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}})$. Also, $\mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}}) \subseteq \underline{\mathcal{P}_O}$ follows from $\mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}}) \subseteq \mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}})$. Thus, $\mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}}) \neq \emptyset$ implies that $\underline{\mathcal{P}_O} \neq \emptyset$. Conversely, suppose that $\underline{\mathcal{P}_O} \neq \emptyset$. Let $\widehat{\mathbf{p}} > 0$ with $\widehat{\mathbf{p}} \in \underline{\mathcal{P}_O}$. The sequence $\widehat{\mathbf{p}}^{(n)} = \mathcal{V}(\widehat{\mathbf{p}}^{(n-1)})$, $\widehat{\mathbf{p}}^{(0)} = \widehat{\mathbf{p}}$, is componentwise monotonically decreasing, that is, $\widehat{\mathbf{p}}_k^{(n+1)} \leq \widehat{\mathbf{p}}_k^{(n)}$, for all k. Thus, there exists a limit $\widetilde{\mathbf{p}} \geq 0$ with $\lim_{n\to\infty} \widehat{\mathbf{p}}_k^{(n)} = \widetilde{\mathbf{p}}_k$. We have $\widetilde{\mathbf{p}} \in \underline{\mathcal{P}_O}$ and thus $\widetilde{\mathbf{p}} > 0$. Since

$$\widetilde{\mathbf{p}} = \lim_{n\to\infty} \widehat{\mathbf{p}}^{(n)} = \lim_{n\to\infty} \mathcal{V}(\widehat{\mathbf{p}}^{(n-1)}) = \mathcal{V}(\widetilde{\mathbf{p}}), \tag{73}$$

we can conclude that $\widetilde{\mathbf{p}} \in \mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}})$. □

For $K = 2$ and no self-interference, the interference functions have the special structure (23). It follows (see Theorem 8) that $\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}}) = \mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}}) \neq \emptyset$. The same holds for $K = 3$, as shown in Section 4.3. For $K = 2, 3$, (no self-interference) all boundary points are effectively achievable, that is, $\text{SIR}_k(\mathbf{p}) \geq \gamma_k$, for all k.

For $K = 4$, we can have $\mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}}) = \emptyset$ and $\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}}) \neq \emptyset$, thus $\bigcap_{l=1}^{\infty} \mathcal{V}^l(\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}})) = \emptyset$. This can be shown by an example. Consider

$$\boldsymbol{\Psi} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \end{bmatrix}, \tag{74}$$

where $0 < b < 1$ and $\boldsymbol{\Gamma}_{\text{Q}} = \text{diag}\{[1, 1, 1, 1]\}$, then there exists a vector $\mathbf{p}^* > 0$ such that

$$[\boldsymbol{\Gamma}_{\text{Q}} \boldsymbol{\Psi} \mathbf{p}^*]_k \leq \mathbf{p}_k^*, \tag{75}$$

where strict inequality holds for the last two components. But this inequality cannot be fulfilled with equality since the second block is isolated (no other blocks in the same row) and has a spectral radius smaller than one (see also [23] for more details). The sequence $(\mathbf{p}^*)^{(n)} = \mathcal{V}((\mathbf{p}^*)^{(n-1)})$ converges to a limit $\lim_{n\to\infty} (\mathbf{p}^*)^{(n)} = [1, 1, 0, 0]$, thus the set $\mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}})$ is empty.

An example for the case $\mathcal{P}_O(\boldsymbol{\Gamma}_{\text{Q}}) = \mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}})$ is the interference function $\mathcal{I}_k(\mathbf{p}, \epsilon)$, as defined in (31). Since $\mathcal{I}_k(\mathbf{p}, \epsilon)$ is strictly monotonically increasing in each power component, we always have $\mathbf{p}^{(\epsilon)} \in \mathcal{P}_E(\boldsymbol{\Gamma}_{\text{Q}})$. This corresponds to a system where all users are coupled.

5. MONOTONICITY PROPERTIES

We have shown that the system of (42) is connected with the SIR balancing problem (7). The existence of a nonnegative solution has been shown in Theorem 6. The following questions remain open.

(i) When is (42) fulfilled by a strictly positive vector $\mathbf{p}^* > 0$?
(ii) When is the solution unique?

With the general model based on (A1)–(A4), it was not possible to provide general answers to these questions (except for $K = 2, 3$ and no self interference). Thus, in the following we consider cases where $\mathcal{I}_k$ has certain monotonicity properties. We consider three different scenarios (M1)–(M3).

(M1) Let $\mathbf{p} \geq 0$ be arbitrary and $\mathbf{p}^* \geq \mathbf{p}$, then for all l with $\mathbf{p}_l^* > \mathbf{p}_l$, we have

$$\mathcal{I}_k(\mathbf{p}^*) > \mathcal{I}_k(\mathbf{p}) \quad \forall k \neq l. \tag{76}$$

(M2) Let $\mathbf{p} \geq 0$ be arbitrary and $\mathbf{p}^* \geq \mathbf{p}$, $\mathbf{p}^* > 0$, then for all l with $\mathbf{p}_l = 0$, we have

$$\mathcal{I}_k(\mathbf{p}^*) > \mathcal{I}_k(\mathbf{p}) \quad \forall k \neq l. \tag{77}$$

(M3) Let $\mathbf{p} > 0$ be arbitrary and $\mathbf{p}^* \geq \mathbf{p}$, then for all l with $\mathbf{p}_l^* > \mathbf{p}_l$, we have

$$\mathcal{I}_k(\mathbf{p}^*) > \mathcal{I}_k(\mathbf{p}) \quad \forall k \neq l. \tag{78}$$

Property (M1) is the most general property. It means that decreasing one users' power always reduces the interference experienced by all other users. Property (M2) says that by switching off one user, we strictly reduce the interference of all other users. (M2) is included in (M1), but not vice versa. Finally, property (M3) is similar to M1, but less restrictive since it is only required for positive powers $\mathbf{p} > 0$.

Theorem 13. *Let $\mathcal{I}_k$ have the property (M1). If $\bar{\mathbf{p}} \geq 0$, $\bar{\mathbf{p}} \neq 0$, fulfills*

$$\gamma_k \mathcal{I}_k(\bar{\mathbf{p}}) = C(\mathbf{\Gamma}_{\mathrm{Q}})\bar{\mathbf{p}}_k, \quad 1 \leq k \leq K, \tag{79}$$

then $\bar{\mathbf{p}} > 0$.

Proof. From Theorem 6, we know that there always exists a nontrivial solution $\bar{\mathbf{p}} \geq 0$. Thus, there exists an index k such that $\bar{\mathbf{p}}_k > 0$. Property (M1) implies that $\mathcal{I}_l(\bar{\mathbf{p}}) > \mathcal{I}_l(\mathbf{0}) = 0$, for all $l \neq k$. From (79), it follows that $\bar{\mathbf{p}}_l > 0$, for all $l \neq k$, and thus $\bar{\mathbf{p}} > 0$. □

The theorem shows that if the interference function is characterized by (M1), then each power allocation which satisfies (79) must be strictly positive. The following corollary shows uniqueness of this solution.

Corollary 14. *If $\mathcal{I}_k$ has the monotonicity property (M1), then there always exists exactly one vector $\bar{\mathbf{p}} > 0$, with $\|\bar{\mathbf{p}}\|_1 = 1$, such that* (79) *holds.*

Proof. This follows from Thereoms 6, 13, and 11. □

Theorem 15. *Let $\mathcal{I}_k$ have the property (M2), and suppose that there exists a $\mathbf{p}^* > 0$ such that*

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*) = C(\mathbf{\Gamma}_{\mathrm{Q}})\mathbf{p}_k^*, \quad 1 \leq k \leq K. \tag{80}$$

Then, all vectors $\bar{\mathbf{p}}$ which fulfill (80) *are strictly positive, that is,* $\bar{\mathbf{p}} > 0$.

Proof. It has been shown (Theorem 6) that there exists a $\bar{\mathbf{p}} \geq 0$ which fulfills (80). It remains to show that $\bar{\mathbf{p}}$ is strictly positive under the assumption that there exists a $\mathbf{p}^* > 0$ which fulfills (80).

The proof is by contradiction. Suppose that $\bar{\mathbf{p}}_l = 0$ for an arbitrary index l. Power allocations which fulfill (80) can be scaled arbitrarily, thus we can assume that $\mathbf{p}^* \geq \bar{\mathbf{p}}$. Thus, there exists a $k_0 \neq l$, with $0 < \mathbf{p}_{k_0}^* = \bar{\mathbf{p}}_{k_0}$, such that

$$C(\mathbf{\Gamma}_{\mathrm{Q}}) = \frac{\gamma_{k_0}\mathcal{I}_{k_0}(\bar{\mathbf{p}})}{\bar{\mathbf{p}}_{k_0}} = \frac{\gamma_{k_0}\mathcal{I}_{k_0}(\bar{\mathbf{p}})}{\mathbf{p}_{k_0}^*} < \frac{\gamma_{k_0}\mathcal{I}_{k_0}(\mathbf{p}^*)}{\mathbf{p}_{k_0}^*} = C(\mathbf{\Gamma}_{\mathrm{Q}}). \tag{81}$$

The inequality follows from (M2) and the assumption $\bar{\mathbf{p}}_l = 0$. From this contradiction, we can conclude that $\bar{\mathbf{p}}_l > 0$, for all l. □

Theorem 16. *Let $\mathcal{I}_k$ be characterized by monotonicity properties (M2) and (M3). Suppose that there exists a $\mathbf{p}^* > 0$ such that* (80) *holds. Then, this solution is unique, that is, there is no other vector $\bar{\mathbf{p}} \geq 0$ which fulfills* (80).

Proof. It is known from Theorem 15 that the existence of one solution $\mathbf{p}^* > 0$ would imply that $\bar{\mathbf{p}} > 0$ for every other solution $\bar{\mathbf{p}}$ that fulfills (80). We can scale $\mathbf{p}^*$ such that $\bar{\mathbf{p}} \geq \mathbf{p}^*$ and with (M3) this would lead to a contradiction. □

6. MATRIX-BASED INTERFERENCE FUNCTIONS

In the previous section, we assumed *arbitrary* interference functions $\mathcal{I}_k(\mathbf{p})$, fulfilling the axioms (A1)–(A3). Now, we assume that $\mathcal{I}_k(\mathbf{p})$ has a specific structure. Namely, we consider the practically relevant case where the power coupling between the links is modelled by a nonnegative matrix $\mathbf{\Psi}(z)$, and the interference experienced by the kth link can be expressed as $[\mathbf{\Psi}(z)\mathbf{p}]_k$. Here, the parameter z stands for some receive strategy. Unless otherwise stated, $\mathbf{\Psi}(z)$ can include self-interference (nonzero entries on the main diagonal).

In order to keep the results as general as possible, we do not make any assumption regarding the nature of z, except that $z = \{z_1, \ldots, z_K\}$, where z_k stands for the receive strategy employed by the kth link. The strategy z_k is chosen from the compact set $\mathcal{Z}_k$, which contains all possible receive strategies for the kth user. For a given power allocation, each z_k can be optimized independently. The overall receive strategy is $z \in \mathcal{Z}$, where $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 \times \cdots \times \mathcal{Z}_K$ (Cartesian product). The coupling matrix $\mathbf{\Psi}(z)$ is assumed to be continuous on $\mathcal{Z}$.

We consider a special type of interference function, namely

$$\mathcal{I}_k(\mathbf{p}) = \min_{z_k \in \mathcal{Z}_k} \sum_{l=1}^{K} \mathbf{\Psi}_{kl}(z_k)\mathbf{p}_l. \tag{82}$$

That is, for any given power allocation $\mathbf{p}$, the receivers are adaptively adjusted so as to minimize the interference of the respective user, which is equivalent to maximizing $\mathrm{SIR}_k(\mathbf{p})$.

Note that the model (82) is an important special case of the more general axiomatic model used in the previous section. It can easily be verified that the interference functions (82) fulfill the axioms (A1)–(A3).

In the following, we will show that also (A4) (continuity) is always fulfilled. Then, we study the relationship between the SIR balancing problem and eigenvalue optimization. Additional properties will be shown under the assumption of irreducibility. Most of the properties, which have been derived for the general axiomatic model in Section 4, are strict in a sense that the conditions cannot be relaxed even for the simple and more restrictive model (82). This will be demonstrated by examples.

6.1. Continuity for a special class of interference function

Continuity of the functions $\mathcal{I}_k(\mathbf{p})$ was already shown in Section 3.1 for the general axiomatic model, except for the boundary of the set $\{\mathbf{p} : \mathbf{p} \geq 0\}$. Now, we show for the special matrix-based interference function (82) (which can be used to model interference for virtually any practical system) that continuity also holds on the boundary.

Theorem 17. *The function $\mathcal{I}_k(\mathbf{p})$, as defined in* (82)*, is continuous for* $\mathbf{p} \geq 0$.

Proof. For $\mathbf{p} > 0$, continuity has already been shown in Theorem 2. Now, consider the sequence $\mathbf{p}^{(n)} \geq 0$, with

$\lim_{n\to\infty} \mathbf{p}^{(n)} = \mathbf{p}$, where $\mathbf{p} \geq 0$ is arbitrary. Let $z(\mathbf{p})$ and $z(\mathbf{p}^{(n)})$ be the receive strategy which minimizes the interference for given $\mathbf{p}$ and $\mathbf{p}^{(n)}$, respectively. Then,

$$\begin{aligned}\mathcal{I}_k(\mathbf{p}) - \mathcal{I}_k(\mathbf{p}^{(n)}) &= \sum_l \mathbf{\Psi}_{kl}(z(\mathbf{p}))\mathbf{p}_l - \sum_l \mathbf{\Psi}_{kl}(z(\mathbf{p}^{(n)}))\mathbf{p}_l^{(n)} \\ &\geq \sum_l \mathbf{\Psi}_{kl}(z(\mathbf{p}))(\mathbf{p}_l - \mathbf{p}_l^{(n)}), \\ \mathcal{I}_k(\mathbf{p}) - \mathcal{I}_k(\mathbf{p}^{(n)}) &\leq \sum_l \mathbf{\Psi}_{kl}(z(\mathbf{p}^{(n)}))(\mathbf{p}_l - \mathbf{p}_l^{(n)}).\end{aligned} \tag{83}$$

Defining

$$\begin{aligned}K_1^{(n)} &= \sum_l \mathbf{\Psi}_{kl}(z(\mathbf{p}))\,|\mathbf{p}_l - \mathbf{p}_l^{(n)}|, \\ K_2^{(n)} &= \sum_l \mathbf{\Psi}_{kl}(z(\mathbf{p}^{(n)}))\,|\mathbf{p}_l - \mathbf{p}_l^{(n)}|,\end{aligned} \tag{84}$$

it follows with (83) that

$$|\,\mathcal{I}_k(\mathbf{p}) - \mathcal{I}_k(\mathbf{p}^{(n)})\,| \leq \max\left(K_1^{(n)}, K_2^{(n)}\right). \tag{85}$$

Since $\mathcal{Z}$ is a compact set, we have

$$C_k := \max_{z\in\mathcal{Z}} \sum_l \mathbf{\Psi}_{kl}(z) < +\infty. \tag{86}$$

Thus, $K_1^{(n)}$ can be upper bounded as follows:

$$\begin{aligned}K_1^{(n)} &\leq \left(\sum_l \mathbf{\Psi}_{kl}(z(\mathbf{p}))\right) \max_{1\leq r\leq K} |\mathbf{p}_r - \mathbf{p}_r^{(n)}| \\ &\leq C_k \cdot ||\mathbf{p} - \mathbf{p}^{(n)}||_\infty.\end{aligned} \tag{87}$$

Similarly, we have

$$K_2^{(n)} \leq C_k \cdot ||\mathbf{p} - \mathbf{p}^{(n)}||_\infty. \tag{88}$$

Thus,

$$|\,\mathcal{I}_k(\mathbf{p}) - \mathcal{I}_k(\mathbf{p}^{(n)})\,| \leq C_k \cdot ||\mathbf{p} - \mathbf{p}^{(n)}||_\infty. \tag{89}$$

Since $\mathbf{p}^{(n)} \to \mathbf{p}$, we have $\lim_{n\to\infty} |\mathcal{I}_k(\mathbf{p}) - \mathcal{I}_k(\mathbf{p}^{(n)})| \leq 0$, thus

$$\lim_{n\to\infty} \mathcal{I}_k(\mathbf{p}^{(n)}) = \mathcal{I}_k(\mathbf{p}), \tag{90}$$

which means that $\mathcal{I}_k$ is continuous for all $\mathbf{p} \geq 0$. □

Thus, (A1)–(A4) are fulfilled and we can apply the analytical results of Section 4. In particular, we know from Theorem 6 that there always exists a vector $\mathbf{p}^* \geq 0$, $\mathbf{p}^* \neq \mathbf{0}$, such that

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*) = \min_{z_k\in\mathcal{Z}_k} [\mathbf{\Gamma}_Q \mathbf{\Psi}(z)\mathbf{p}^*]_k = C(\mathbf{\Gamma}_Q)\mathbf{p}_k^* \quad \forall k. \tag{91}$$

In the following, we study how the balanced optimum $C(\mathbf{\Gamma}_Q)$ can be described by an eigenvalue optimization problem.

6.2. Relationship with eigenvalue optimization

Next, we study the aspect of feasibility. With the results of Section 4, we know that feasibility is completely characterized by $C(\mathbf{\Gamma}_Q)$, which is the optimum of the min-max balancing problem (7) under the assumption of matrix-based interference functions (82). Optimization is over all positive power allocations.

Another meaningful way of characterizing feasibility is by means of the spectral radius of the coupling matrix $\mathbf{\Psi}(z)$. If $\mathbf{\Psi}(z)$ is irreducible and z is a fixed parameter, then it is known that targets $\mathbf{\Gamma}_Q$ are feasible if and only if $\rho(\mathbf{\Gamma}_Q\mathbf{\Psi}(z)) \leq 1$ (see, e.g., [4, 6, 26]).

However, the problem at hand differs from this classical problem formulation, in that we consider the joint optimization of transmission powers $\mathbf{p}$ and the receive strategy z. The coupling matrix $\mathbf{\Psi}(z)$ can depend on the parameter z in such a way that interference terms are cancelled or nulled out. Thus, $\mathbf{\Psi}(z)$ can become *reducible*, which means that the system becomes partly or even fully decoupled and the Perron-Frobenius theorem for irreducible matrices cannot be applied.

Thus, it is desirable to have a more general notion of feasibility, which is not based on the assumption of irreducibility. If we only assume *nonnegativeness*, then $\mathbf{\Gamma}_Q$ is feasible for a given z, if and only if there exists a $\mathbf{p} > 0$ such that $\max_k [\mathbf{\Gamma}_Q\mathbf{\Psi}(z)\mathbf{p}]_k/\mathbf{p}_k \leq 1$. The infimum over all $\mathbf{p} > 0$ equals the spectral radius ρ of the weighted coupling matrix $\mathbf{\Gamma}_Q\mathbf{\Psi}(z)$. This follows from the following Collatz-Wielandt-type characterization [27]:

$$\rho(\mathbf{\Gamma}_Q\mathbf{\Psi}(z)) = \inf_{\mathbf{p}>0}\left(\max_{1\leq k\leq K} \frac{[\mathbf{\Gamma}_Q\mathbf{\Psi}(z)\mathbf{p}]_k}{\mathbf{p}_k}\right). \tag{92}$$

By taking the infimum over all possible receive strategies $z \in \mathcal{Z}$, we obtain a necessary and sufficient condition for the feasibility of $\mathbf{\Gamma}_Q$:

$$\inf_{z\in\mathcal{Z}} \rho(\mathbf{\Gamma}_Q\mathbf{\Psi}(z)) \leq 1. \tag{93}$$

Note that the optimum (93) is obtained by optimizing over z, while the SIR balancing optimum

$$C(\mathbf{\Gamma}_Q) = \inf_{\mathbf{p}>0}\left(\max_{1\leq k\leq K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{\mathbf{p}_k}\right) \tag{94}$$

is obtained by optimization over $\mathbf{p}$. The min-max optimum (94) implicitly includes the optimization over the receivers by considering the special interference functions (82), while the right-hand side of (92)is for one specific choice of z.

Both strategies (93) and (94) are meaningful ways to define the feasible region. If $\mathbf{\Psi}(z)$ is irreducible, then it follows from the Perron-Frobenius theorem [28] that both strategies provide equivalent indicators for feasibility. The following theorem shows that this can be extended to *general* nonnegative matrices $\mathbf{\Psi}(z) \geq 0$.

Theorem 18. *Let $\mathbf{\Psi}(z)$ be nonnegative (not necessarily irreducible), then*

$$C(\mathbf{\Gamma}_Q) = \inf_{z\in\mathcal{Z}} \rho(\mathbf{\Gamma}_Q\mathbf{\Psi}(z)). \tag{95}$$

Proof. Since $\mathcal{I}_k$ is designed to minimize the interference, we have for every $z \in \mathcal{Z}$,

$$\max_{1\le k\le K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{\mathbf{p}_k} \le \max_{1\le k\le K} \frac{[\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}(z)\mathbf{p}]_k}{\mathbf{p}_k}. \tag{96}$$

Taking the infimum over $\mathbf{p} > 0$ on both sides and using (92) and (94), we have $C(\boldsymbol{\Gamma}_{\mathrm{Q}}) \le \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}(z))$ for any z, and thus

$$C(\boldsymbol{\Gamma}_{\mathrm{Q}}) \le \inf_{z\in\mathcal{Z}} \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}(z)). \tag{97}$$

Assume that we have an arbitrary $\epsilon > 0$, then it can be seen from (94) that there exists a $\mathbf{p}^{(\epsilon)} > 0$ such that

$$\max_{1\le k\le K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p}^{(\epsilon)})}{\mathbf{p}_k^{(\epsilon)}} \le C(\boldsymbol{\Gamma}_{\mathrm{Q}}) + \epsilon. \tag{98}$$

This inequality holds for all indices k. There exists a $z^{(\epsilon)}$ such that $\mathcal{I}_k(\mathbf{p}^{(\epsilon)}) = [\boldsymbol{\Psi}(z^{(\epsilon)})\mathbf{p}^{(\epsilon)}]_k$, $k \in \{1,2,\dots,K\}$. Thus,

$$\max_{1\le k\le K} \frac{\gamma_k \left[\boldsymbol{\Psi}(z^{(\epsilon)})\mathbf{p}^{(\epsilon)}\right]_k}{\mathbf{p}_k^{(\epsilon)}} \le C(\boldsymbol{\Gamma}_{\mathrm{Q}}) + \epsilon. \tag{99}$$

Consequently,

$$\inf_{\mathbf{p}>0}\left(\max_{1\le k\le K} \frac{\gamma_k [\boldsymbol{\Psi}(z^{(\epsilon)})\mathbf{p}]_k}{\mathbf{p}_k}\right) \le C(\boldsymbol{\Gamma}_{\mathrm{Q}}) + \epsilon. \tag{100}$$

Combining (92) and (100), we have $\rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}(z^{(\epsilon)})) \le C(\boldsymbol{\Gamma}_{\mathrm{Q}}) + \epsilon$, and thus

$$\inf_{z\in\mathcal{Z}} \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}(z)) \le C(\boldsymbol{\Gamma}_{\mathrm{Q}}) + \epsilon. \tag{101}$$

This holds for all $\epsilon > 0$, thus

$$\inf_{z\in\mathcal{Z}} \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}(z)) \le C(\boldsymbol{\Gamma}_{\mathrm{Q}}). \tag{102}$$

Comparison with (97) shows the desired result. □

6.3. Structure properties of matrices

Now, we show additional properties under the assumption that the coupling matrix has a specific structure. To this end, we can rearrange the user indices such that $\boldsymbol{\Psi}$ has the following block form [22]:

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}^{(1,1)} & 0 & \dots & 0 \\ \boldsymbol{\Psi}^{(2,1)} & \boldsymbol{\Psi}^{(2,2)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \boldsymbol{\Psi}^{(N,1)} & \boldsymbol{\Psi}^{(N,2)} & \dots & \boldsymbol{\Psi}^{(N,N)} \end{bmatrix}, \tag{103}$$

where each block $\boldsymbol{\Psi}^{(m,n)}$ is irreducible. We assume that each user receives interference from at least one other user, thus each block has at least size two. If $\boldsymbol{\Psi}$ is irreducible, then this is just a special case of (103), where the matrix simply consists of one single block.

Definition 2. A diagonal block $\boldsymbol{\Psi}^{(n)} := \boldsymbol{\Psi}^{(n,n)}$ is called *isolated* if and only if $\boldsymbol{\Psi}^{(n,r)} = 0$ for $r = 1,2,\dots,n-1,n+1,\dots,N$.

By permuting the indices, the matrix can be arranged such that the isolated blocks are the first blocks on the main diagonal. Without loss of generality, we will assume that $\boldsymbol{\Psi}$ is always arranged in this block normal form.

Definition 3. A matrix $\boldsymbol{\Psi}$ is *block irreducible*, if and only if it has the following block-diagonal structure:

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{\Psi}^{(N)} \end{bmatrix} \tag{104}$$

and each block is irreducible.

Definition 4. For any diagonal block $\boldsymbol{\Psi}^{(n)}$,

$$\rho(\boldsymbol{\Gamma}_{\mathrm{Q}}^{(n)}\boldsymbol{\Psi}^{(n)}) \le \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}). \tag{105}$$

The block is called *maximal* if and only if $\rho(\boldsymbol{\Gamma}_{\mathrm{Q}}^{(n)}\boldsymbol{\Psi}^{(n)}) = \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi})$. There is always at least one maximal diagonal block.

In the following, we discuss some properties which help to understand the connection between the general axiom-based SIR balancing theory and the matrix theory. To this end, consider a power allocation $\bar{\mathbf{p}} > 0$ such that

$$\gamma_k \mathcal{I}_k(\bar{\mathbf{p}}) = C(\boldsymbol{\Gamma}_{\mathrm{Q}})\bar{\mathbf{p}}_k, \quad 1 \le k \le K, \tag{106}$$

then the min-max optimum $C(\boldsymbol{\Gamma}_{\mathrm{Q}})$ equals the max-min optimum $c(\boldsymbol{\Gamma}_{\mathrm{Q}})$. This is because

$$c(\boldsymbol{\Gamma}_{\mathrm{Q}}) = \sup_{\mathbf{p}>0}\left(\min_{1\le k\le K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{\mathbf{p}_k}\right) \ge \min_{1\le k\le K} \frac{\gamma_k \mathcal{I}_k(\bar{\mathbf{p}})}{\bar{\mathbf{p}}_k} = C(\boldsymbol{\Gamma}_{\mathrm{Q}}), \tag{107}$$

where the last step follows from (106). From Theorem 4, we know that $C(\boldsymbol{\Gamma}_{\mathrm{Q}}) \ge c(\boldsymbol{\Gamma}_{\mathrm{Q}})$, thus the inequality can only be fulfilled with equality, that is, $c(\boldsymbol{\Gamma}_{\mathrm{Q}}) = C(\boldsymbol{\Gamma}_{\mathrm{Q}})$. Note that the converse does not need to hold. That is, $C(\boldsymbol{\Gamma}_{\mathrm{Q}}) = c(\boldsymbol{\Gamma}_{\mathrm{Q}})$ does not necessarily imply (106). It is required that the structure of $\boldsymbol{\Psi}$ is such that $c(\boldsymbol{\Gamma}_{\mathrm{Q}}) = C(\boldsymbol{\Gamma}_{\mathrm{Q}})$ holds for *all* $\boldsymbol{\Gamma}_{\mathrm{Q}} > 0$. This is specified by the following theorem. For a characterization of $\boldsymbol{\Psi}$, such that (107) holds for some *fixed* $\boldsymbol{\Gamma}$, we refer to [29], where also the equivalence of max-min and min-max fair resource allocations and the relationship with the QoS feasible region are investigated.

Theorem 19. *The following statements are equivalent:*

(a) $c(\boldsymbol{\Gamma}_{\mathrm{Q}}) = C(\boldsymbol{\Gamma}_{\mathrm{Q}})$ *for all* $\boldsymbol{\Gamma}_{\mathrm{Q}} > 0$,
(b) $\boldsymbol{\Psi}$ *is irreducible,*
(c) *for every* $\boldsymbol{\Gamma} > 0$, *there exists a* $\mathbf{p}_\Gamma > 0$ *such that*

$$\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}\mathbf{p}_\Gamma = C(\boldsymbol{\Gamma}_{\mathrm{Q}})\mathbf{p}_\Gamma. \tag{108}$$

Proof. If $\boldsymbol{\Psi}$ is irreducible, then we know from the Perron-Frobenius theorem [28] that there always exists a $\mathbf{p} > 0$ such that (108) is fulfilled, and thus $c(\boldsymbol{\Gamma}_{\mathrm{Q}}) = C(\boldsymbol{\Gamma}_{\mathrm{Q}})$. Thus, (b) implies (a) and (c).

We now prove (a) ⇒ (b) by contradiction. Suppose that $c(\boldsymbol{\Gamma}_{\mathrm{Q}}) = C(\boldsymbol{\Gamma}_{\mathrm{Q}})$ for all $\boldsymbol{\Gamma}_{\mathrm{Q}}$, but $\boldsymbol{\Psi}$ is not irreducible. Then $\boldsymbol{\Psi}$ can be arranged in block normal form (103). Since $C(\boldsymbol{\Gamma}_{\mathrm{Q}}) = \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi})$, we have

$$C(\boldsymbol{\Gamma}_{\mathrm{Q}}) = \max_{1\leq n\leq N} \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}^{(n)}\boldsymbol{\Psi}^{(n)}), \tag{109}$$

where $\boldsymbol{\Gamma}_{\mathrm{Q}}^{(n)}\boldsymbol{\Psi}^{(n)}$ is the nth irreducible diagonal subblock of $\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}$. We have

$$\begin{aligned} c(\boldsymbol{\Gamma}_{\mathrm{Q}}) &= \sup_{\mathbf{p}>0}\left(\min_{1\leq k\leq K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{\mathbf{p}_k}\right) \\ &\leq \sup_{\mathbf{p}>0}\left(\min_{k\in\mathcal{J}^{(n)}} \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{\mathbf{p}_k}\right), \end{aligned} \tag{110}$$

where $\mathcal{J}^{(n)}$ is the index set associated with the nth diagonal block. This holds for all blocks $1 \leq n \leq N$. Since $\boldsymbol{\Psi}$ is not irreducible, there is at least one isolated block. Without loss of generality, we can assume that the users are ordered such that $\boldsymbol{\Psi}^{(1)}$ is the first isolated block. Exploiting the fact that $\boldsymbol{\Psi}^{(1)}$ is isolated, we have

$$\begin{aligned} c(\boldsymbol{\Gamma}_{\mathrm{Q}}) &\leq \sup_{\mathbf{p}>0}\left(\min_{k\in\mathcal{J}^{(1)}} \frac{\gamma_k}{\mathbf{p}_k}\sum_{k=1}^{K}\boldsymbol{\Psi}_{kl}\mathbf{p}_l\right) \\ &= \sup_{\mathbf{p}>0}\left(\min_{k\in\mathcal{J}^{(1)}} \frac{\gamma_k}{\mathbf{p}_k}\sum_{l\in\mathcal{J}^{(1)}}\boldsymbol{\Psi}_{kl}^{(1)}\mathbf{p}_l\right) = \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}^{(1)}\boldsymbol{\Psi}^{(1)}). \end{aligned} \tag{111}$$

Since the assumption (a) holds for all $\boldsymbol{\Gamma}_{\mathrm{Q}} > 0$, we can choose $\boldsymbol{\Gamma}_{\mathrm{Q}}$ such that

$$\rho(\boldsymbol{\Gamma}_{\mathrm{Q}}^{(1)}\boldsymbol{\Psi}^{(1)}) < \max_{n\geq 2} \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}^{(n)}\boldsymbol{\Psi}^{(n)}) = C(\boldsymbol{\Gamma}_{\mathrm{Q}}), \tag{112}$$

which, combined with (111), contradicts the assumption (a).

It remains to show that (c) ⇒ (b). To this end, we can assume that $C(\boldsymbol{\Gamma}_{\mathrm{Q}}) = 1$ without loss of generality. Let (c) be fulfilled, that is, (108) holds for all $\boldsymbol{\Gamma}_{\mathrm{Q}}$, but $\boldsymbol{\Psi}$ is not irreducible. Then, $\boldsymbol{\Psi}$ can be arranged in block normal form (103) with at least one isolated subblock. Since (108) is assumed to hold for all $\boldsymbol{\Gamma}_{\mathrm{Q}}$, we can choose $\boldsymbol{\Gamma}_{\mathrm{Q}}$ such that one isolated subblock has a spectral radius smaller than one. This would rule out the existence of a positive eigenvector $\mathbf{p}' > 0$ such that $\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}\mathbf{p}' = \mathbf{p}'$, thus it would contradict the assumption (c). Hence, $\boldsymbol{\Psi}$ is irreducible. □

Recall that the existence of a vector $\bar{\mathbf{p}} > 0$ such that

$$\gamma_k \mathcal{I}_k(\bar{\mathbf{p}}) \leq C(\boldsymbol{\Gamma}_{\mathrm{Q}})\bar{\mathbf{p}}_k, \quad 1 \leq k \leq K, \tag{113}$$

does not imply that (113) can be fulfilled with equality. That is, a boundary point $\boldsymbol{\Gamma}_{\mathrm{Q}}$ with $C(\boldsymbol{\Gamma}_{\mathrm{Q}}) = 1$ might be effectively achievable, that is, there exists a power allocation $\mathbf{p}^* > 0$ such that $\mathrm{SIR}_k(\mathbf{p}^*) \geq \gamma_k$, for all k. But there does not need to exist an allocation such that the targets γ_k are achieved with equality (see also Section 4.6).

Whether or not $\boldsymbol{\Gamma}_{\mathrm{Q}}$ can be achieved with equality depends on the structure of the coupling matrix $\boldsymbol{\Psi}$ (see definitions at the beginning of Section 6). In particular, $\mathrm{SIR}_k(\mathbf{p}^*) \geq \gamma_k$, for all k, can be fulfilled if and only if the set of maximal blocks is a subset of the isolated blocks. In this case, $c(\boldsymbol{\Gamma}_{\mathrm{Q}}) < C(\boldsymbol{\Gamma}_{\mathrm{Q}})$ holds. But $c(\boldsymbol{\Gamma}_{\mathrm{Q}}) = C(\boldsymbol{\Gamma}_{\mathrm{Q}})$ requires that the maximal blocks coincide with the isolated blocks.

6.4. Examples for strictness

One could expect that under the restriction to a simple matrix model of the form (82), the results of Section 4 could be extended by showing additional properties. However, this is not the case, as will be shown in the following. All results in Section 4 also apply to the special matrix-based functions (82). In this sense, the results are strict.

Consider a boundary point $\boldsymbol{\Gamma}_{\mathrm{Q}}$ and the weighted coupling matrix

$$\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{A} & 0 \\ 0 & \boldsymbol{B} \end{bmatrix}. \tag{114}$$

The spectral radius is $\rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}) = \max(\rho(\boldsymbol{A}), \rho(\boldsymbol{B}))$. Assume that the irreducible subblocks are chosen such that only the first block is maximal, that is, $\rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}) = \rho(\boldsymbol{A}) > \rho(\boldsymbol{B})$.

The second subblock is not maximal, thus there does not exist a strictly positive eigenvector to $\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}$. But the irreducible subblocks $\boldsymbol{A}$ and $\boldsymbol{B}$ have strictly positive dominant right eigenvectors $\mathbf{p}^{(A)}$ and $\mathbf{p}^{(B)}$, respectively (according to the Perron-Frobenius theory). Thus, $\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}$ has nonnegative eigenvectors $\bar{\mathbf{p}}^{(A)} = \left[\begin{smallmatrix}\mathbf{p}^{(A)} \\ 0\end{smallmatrix}\right]$ and $\bar{\mathbf{p}}^{(B)} = \left[\begin{smallmatrix}0 \\ \mathbf{p}^{(B)}\end{smallmatrix}\right]$, which fulfill

$$\begin{aligned} \boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}\bar{\mathbf{p}}^{(A)} &= \rho(\boldsymbol{A})\bar{\mathbf{p}}^{(A)}, \\ \boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}\bar{\mathbf{p}}^{(B)} &= \rho(\boldsymbol{B})\bar{\mathbf{p}}^{(B)}. \end{aligned} \tag{115}$$

From (92), we know that the optimum $C(\boldsymbol{\Gamma}_{\mathrm{Q}})$ (optimization over $\mathbf{p} > 0$) equals the spectral radius $\rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi})$. The example shows that if we replace the constraint $\mathbf{p} > 0$ by $\mathbf{p} \geq 0$, that is, if users are allowed to switch off, then a smaller level (associated with a larger QoS region) can be achieved. This is exactly what is stated by Theorem 10. That is, $\boldsymbol{\Gamma}_{\mathrm{Q}}\boldsymbol{\Psi}$ has two different eigenvectors $\mathbf{p} \geq 0$ with different associated eigenvalues, depending on which user is switched off.

As another example, consider the matrix

$$\boldsymbol{\Gamma}_{\mathrm{Q}}\bar{\boldsymbol{\Psi}} = \begin{bmatrix} \boldsymbol{A} & 0 \\ \boldsymbol{C} & \boldsymbol{B} \end{bmatrix}. \tag{116}$$

Again, assume that $\rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\bar{\boldsymbol{\Psi}}) = \rho(\boldsymbol{A}) > \rho(\boldsymbol{B})$. The only difference to the previous example is the existence of the off-diagonal block $\boldsymbol{C}$, which means that the subsystem $\boldsymbol{B}$ receives interference from the subsystem $\boldsymbol{A}$.

Example (116) has a special structure, namely the set of isolated blocks (block $\boldsymbol{A}$) coincides with the set of maximal blocks. It can be shown that only then, there exists a $\bar{\mathbf{p}} > 0$ such that

$$\boldsymbol{\Gamma}_{\mathrm{Q}}\bar{\boldsymbol{\Psi}}\bar{\mathbf{p}} = \rho(\boldsymbol{\Gamma}_{\mathrm{Q}}\bar{\boldsymbol{\Psi}})\bar{\mathbf{p}}. \tag{117}$$

As in the previous example, the first user can be switched off, thus there also exists a nonnegative eigenvector $\left[\begin{smallmatrix} 0 \\ \mathbf{p}^{(2)} \end{smallmatrix} \right]$ with an eigenvalue $\rho(\boldsymbol{B})$.

To conclude, if there exists a strictly positive solution $\bar{\mathbf{p}} > 0$ such that $\mathrm{SIR}(\bar{\mathbf{p}}) = \gamma_k$, for all k, then Theorem 11 states that this optimum is achieved exactly by the optimization approach (7). The only way to achieve a smaller value is to switch off one or more users.

6.5. Connecting optimal power allocation with eigenvectors

It was shown in Section 6.2 that the min-max optimum $C(\mathbf{\Gamma}_{\mathrm{Q}})$ is equivalently characterized by the eigenvalue problem (93). The set $\mathcal{Z}(\mathbf{\Gamma}_{\mathrm{Q}})$ contains the optimal receive strategies which minimize the spectral radius $\rho(\mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi}(z))$.

The optimum can also be characterized with the interference function $\mathcal{I}_k$, as defined in (82). There always exists a $\bar{\mathbf{p}} \geq 0$ such that

$$\gamma_k \mathcal{I}_k(\bar{\mathbf{p}}) = C(\mathbf{\Gamma}_{\mathrm{Q}})\bar{\mathbf{p}}_k, \quad 1 \leq k \leq K. \tag{118}$$

For this solution $\bar{\mathbf{p}}$, there exists a $z_{\bar{\mathbf{p}}}$, defined by the relationship (82). Since the optimum $C(\mathbf{\Gamma}_{\mathrm{Q}})$ in (118) is achieved by all links, we can conclude that the solution $z_{\bar{\mathbf{p}}}$ is also optimal with respect to the eigenvalue problem, that is, $z_{\bar{\mathbf{p}}} \in \mathcal{Z}(\mathbf{\Gamma}_{\mathrm{Q}})$.

This optimal receive strategy fulfills

$$\gamma_k \mathcal{I}_k(\bar{\mathbf{p}}) = \gamma_k \sum_{l=1}^{K} \mathbf{\Psi}_{kl}(z_{\bar{\mathbf{p}}})\bar{\mathbf{p}}_l. \tag{119}$$

Comparison with (118) reveals that the vector $\bar{\mathbf{p}}$ is the right principal eigenvector of the matrix $\mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi}(z_{\bar{\mathbf{p}}})$. The optimum $C(\mathbf{\Gamma}_{\mathrm{Q}})$ is the associated eigenvalue.

An interesting question is whether each $\hat{z} \in \mathcal{Z}(\mathbf{\Gamma}_{\mathrm{Q}})$ is a solution of the SIR balancing problem (7). An answer is provided by the following theorem.

Theorem 20. *The optimizer $\hat{z} \in \mathcal{Z}(\mathbf{\Gamma}_{\mathrm{Q}})$ solves the min-max problem (7) if and only if the matrix $\mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi}(\hat{z})$ has a positive eigenvector $\hat{\mathbf{p}} > 0$, associated with an eigenvalue $C(\mathbf{\Gamma}_{\mathrm{Q}})$, such that $\mathcal{I}_k(\hat{\mathbf{p}}) = \sum_{l=1}^{K} \mathbf{\Psi}_{kl}(\hat{z})\hat{\mathbf{p}}_l$.*

Proof. For each $\hat{\mathbf{p}}$ which solves (7), it has been shown that there exists a $\hat{z} \in \mathcal{Z}(\mathbf{\Gamma}_{\mathrm{Q}})$ such that $\hat{\mathbf{p}}$ is the right eigenvector of $\mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi}(\hat{z})$, with eigenvalue $C(\mathbf{\Gamma}_{\mathrm{Q}})$, and $\mathcal{I}_k(\hat{\mathbf{p}}) = [\mathbf{\Psi}(\hat{z})\hat{\mathbf{p}}]_k$ is fulfilled.

Conversely, assume that this characterization is fulfilled, then we can conclude that $\gamma_k \mathcal{I}_k(\hat{\mathbf{p}}) = [\mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi}(\hat{z})\hat{\mathbf{p}}]_k = C(\mathbf{\Gamma}_{\mathrm{Q}})\hat{\mathbf{p}}_k$. □

6.6. Continuity behavior of the functions C and c

It was shown that the min-max optimum $C(\mathbf{\Gamma}_{\mathrm{Q}})$, as defined in (7), and the max-min optimum $c(\mathbf{\Gamma}_{\mathrm{Q}})$, as defined in (24), can be equivalent under certain conditions. Such a behavior is desirable. For example, equivalence was used in [19] to derive upper/lower bounds which control the convergence behavior of an iterative algorithm for joint beamforming and power control.

The value $C(\mathbf{\Gamma}_{\mathrm{Q}})$ is closely linked to the problem of SIR balancing and it has some nice properties. In particular, $C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon)$, as defined in (32), is continuous with respect to ϵ, monotonically decreasing for $\epsilon \to 0$, and converges towards $C(\mathbf{\Gamma}_{\mathrm{Q}})$.

The same behavior does not hold for $c(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon)$, defined as

$$c(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon) = \sup_{\mathbf{p}>0} \min_{1 \leq k \leq K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p}, \epsilon)}{\mathbf{p}_k}. \tag{120}$$

To illustrate this, consider a coupling matrix

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Psi}^{(1)} & 0 \\ \mathbf{\Psi}^{(1,2)} & \mathbf{\Psi}^{(2)} \end{bmatrix}, \tag{121}$$

such that $\rho(\mathbf{\Gamma}_{\mathrm{Q}}^{(1)}\mathbf{\Psi}^{(1)}) < \rho(\mathbf{\Gamma}_{\mathrm{Q}}^{(2)}\mathbf{\Psi}^{(2)})$. Let $\mathbf{1}$ be the all-one matrix, and

$$\mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi} + \epsilon\mathbf{1} =: \mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi}_{\epsilon}. \tag{122}$$

The matrix $\mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi}_{\epsilon}$ is strictly positive and thus irreducible. Consequently,

$$c(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon) = \rho(\mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi}_{\epsilon}) = C(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon), \tag{123}$$

and thus,

$$\lim_{\epsilon \to 0} c(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon) = \lim_{\epsilon \to 0} \rho(\mathbf{\Gamma}_{\mathrm{Q}}\mathbf{\Psi}_{\epsilon}) = \rho(\mathbf{\Gamma}_{\mathrm{Q}}^{(2)}\mathbf{\Psi}^{(2)}). \tag{124}$$

On the other hand, we have (see (111))

$$c(\mathbf{\Gamma}_{\mathrm{Q}}) \leq \rho(\mathbf{\Gamma}_{\mathrm{Q}}^{(1)}\mathbf{\Psi}^{(1)}) < \rho(\mathbf{\Gamma}_{\mathrm{Q}}^{(2)}\mathbf{\Psi}^{(2)}) = \lim_{\epsilon \to 0} c(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon). \tag{125}$$

Thus, $c(\mathbf{\Gamma}_{\mathrm{Q}}, \epsilon)$ does not converge to $c(\mathbf{\Gamma}_{\mathrm{Q}})$. In this respect, $c(\mathbf{\Gamma}_{\mathrm{Q}})$ is not continuous. Both problems (7) and (24) are equivalent if the coupling matrix is irreducible. Then, $C(\mathbf{\Gamma}_{\mathrm{Q}}) = c(\mathbf{\Gamma}_{\mathrm{Q}})$.

Continuity plays an important role in the presence of error effects, for example when $\mathcal{I}_k(\mathbf{p})$ is only known approximately, then continuity ensures that small changes of $\mathbf{p}$ always have a limited effect on $\mathcal{I}_k(\mathbf{p})$.

6.7. Convexity properties of the interference function

Finally, we show additional convexity properties under the assumption of the special interference function $\mathcal{I}_k$, as defined in (82). To this end, consider the power vector

$$\mathbf{p}(\lambda) = (1 - \lambda)\mathbf{p}^{(1)} + \lambda\mathbf{p}^{(2)}, \tag{126}$$

where $\mathbf{p}^{(1)}, \mathbf{p}^{(2)} > 0$ are arbitrary. Then,

$$\begin{aligned}
\mathcal{I}_k(\mathbf{p}(\lambda)) &= \min_{z_k \in \mathcal{Z}_k} \left((1 - \lambda) \sum_{l=1}^{K} \mathbf{\Psi}_{kl}(z_k)\mathbf{p}_l^{(1)} + \lambda \sum_{l=1}^{K} \mathbf{\Psi}_{kl}(z_k)\mathbf{p}_l^{(2)} \right) \\
&\geq (1 - \lambda) \min_{z_k \in \mathcal{Z}_k} \left(\sum_{l=1}^{K} \mathbf{\Psi}_{kl}(z_k)\mathbf{p}_l^{(1)} \right) \\
&\quad + \lambda \min_{z_k \in \mathcal{Z}_k} \left(\sum_{l=1}^{K} \mathbf{\Psi}_{kl}(z_k)\mathbf{p}_l^{(2)} \right) \\
&= (1 - \lambda)\mathcal{I}_k(\mathbf{p}^{(1)}) + \lambda\mathcal{I}_k(\mathbf{p}^{(2)}).
\end{aligned} \tag{127}$$

Thus, $\mathcal{I}_k$ is jointly concave. Consider the set

$$\mathcal{M}(\alpha) = \{\mathbf{p} \geq 0 : \gamma_k \mathcal{I}_k(\mathbf{p}) \geq \alpha \mathbf{p}_k,\ 1 \leq k \leq K,\ \|\mathbf{p}\|_1 = 1\}. \tag{128}$$

The set $\mathcal{M}(\lambda)$ is a closed, bounded set, and $\mathcal{M}(\lambda) \neq \emptyset$ if $\lambda < c(\mathbf{\Gamma}_Q)$. For $\lambda_2 > \lambda_1$, we have $\mathcal{M}(\lambda_2) \subset \mathcal{M}(\lambda_1)$. For $\lambda > c(\mathbf{\Gamma}_Q)$, we have $\mathcal{M}(\lambda) = \emptyset$. Moreover, $\mathcal{M}(\lambda)$ is a convex set because of the concavity of $\mathcal{I}_k$. We have

$$\bigcap_{\lambda < c(\mathbf{\Gamma}_Q)} \mathcal{M}(\lambda) = \mathcal{M}(c(\mathbf{\Gamma}_Q)) \neq \emptyset. \tag{129}$$

The set $\mathcal{M}(c(\mathbf{\Gamma}_Q))$ is convex since the intersection of convex sets is convex.

Theorem 21. *Let $\mathbf{\Psi}(z)$ be irreducible for all $z \in \mathcal{Z}$. If there exists $\mathbf{p}^* > 0$ with $\mathbf{p}^* \in \mathcal{M}(c(\mathbf{\Gamma}_Q))$, then*

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*) = C(\mathbf{\Gamma}_Q)\mathbf{p}_k^*. \tag{130}$$

Proof. Suppose that there exists a $\mathbf{p}^* > 0$ such that

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*) \geq c(\mathbf{\Gamma}_Q)\mathbf{p}_k^*, \quad 1 \leq k \leq K. \tag{131}$$

Then there exists a receive strategy $z(\mathbf{p}^*)$ such that

$$\mathbf{\Gamma}_Q \mathbf{\Psi}(z(\mathbf{p}^*))\mathbf{p}^* \geq c(\mathbf{\Gamma}_Q)\mathbf{p}^*, \tag{132}$$

and it can even be shown that this inequality is fulfilled with equality for all indices (otherwise it would be possible to further increase the maximum $c(\mathbf{\Gamma}_Q)$). This means that $c(\mathbf{\Gamma}_Q)$ equals the spectral radius $\rho(\mathbf{\Gamma}_Q \mathbf{\Psi}(z(\mathbf{p}^*)))$ and $\mathbf{p}^*$ is the associated Perron eigenvector. We have $\gamma_k \mathcal{I}_k(\mathbf{p}^*) = c(\mathbf{\Gamma}_Q)\mathbf{p}^*$, for all k. From Theorem 11, we know that $c(\mathbf{\Gamma}_Q)$ equals the min-max optimum $C(\mathbf{\Gamma}_Q)$, thus (130) is fulfilled. □

Next, we show how the convexity results can be applied.

Theorem 22. *Let $\mathbf{p}^* \in \mathcal{M}(c(\mathbf{\Gamma}_Q))$ with $\mathbf{p}^* > 0$ and there exists a constant c_1, such that the sequence*

$$\underline{\mathbf{p}}_k^{(n+1)} = \frac{1}{c(\mathbf{\Gamma}_Q)} \cdot \gamma_k \mathcal{I}_k(\underline{\mathbf{p}}^{(n)}), \quad \text{with } \mathbf{p}_k^{(1)} = \frac{\gamma_k \mathcal{I}_k(\mathbf{p}^*)}{c(\mathbf{\Gamma}_Q)}, \tag{133}$$

is bounded by c_1, that is,

$$\max_{1 \leq k \leq K} \mathbf{p}_k^{(n+1)} \leq c_1 \quad \forall n, \tag{134}$$

then $c(\mathbf{\Gamma}_Q) = C(\mathbf{\Gamma}_Q)$ and there exists a $\hat{\mathbf{p}} > 0$ such that

$$C(\mathbf{\Gamma}_Q)\hat{\mathbf{p}}_k = \gamma_k \mathcal{I}_k(\hat{\mathbf{p}}), \quad 1 \leq k \leq K. \tag{135}$$

Proof. Since $\mathbf{p}^* \in \mathcal{M}(c(\mathbf{\Gamma}_Q))$, we have

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*) \geq c(\mathbf{\Gamma}_Q)\mathbf{p}_k^*, \quad 1 \leq k \leq K. \tag{136}$$

Without loss of generality, we can assume that $c(\mathbf{\Gamma}_Q) = 1$, thus $\underline{\mathbf{p}}_k^{(1)} \geq \mathbf{p}_k^* > 0$, $1 \leq k \leq K$. It can be shown that $\underline{\mathbf{p}}_k^{(n+1)} \geq \underline{\mathbf{p}}_k^{(n)}$, $1 \leq k \leq K$, that is, the sequence $\underline{\mathbf{p}}_k^{(n)}$ is monotonically increasing. Since $\underline{\mathbf{p}}_k^{(n)}$ is bounded by c_1, there must exists a $\hat{\mathbf{p}} > 0$ such that $\lim_{n\to\infty} \underline{\mathbf{p}}_k^{(n)} = \hat{\mathbf{p}}_k$. Thus, for all k we have

$$\begin{aligned} \hat{\mathbf{p}}_k &= \lim_{n\to\infty} \underline{\mathbf{p}}_k^{(n+1)} = \frac{1}{c(\mathbf{\Gamma}_Q)} \lim_{n\to\infty} \gamma_k \mathcal{I}_k(\underline{\mathbf{p}}^{(n)}) \\ &= \frac{1}{c(\mathbf{\Gamma}_Q)} \gamma_k \mathcal{I}_k(\hat{\mathbf{p}}). \end{aligned} \tag{137}$$

Here, we have used the continuity of the interference functions. Since $\hat{\mathbf{p}} > 0$, we know from Theorem 21 that $c(\mathbf{\Gamma}_Q) = C(\mathbf{\Gamma}_Q)$. □

Analogously, we can show the following result.

Theorem 23. *Let $\mathbf{p}^* > 0$ such that*

$$\gamma_k \mathcal{I}_k(\mathbf{p}^*) \leq C(\mathbf{\Gamma}_Q)\mathbf{p}_k^*, \quad 1 \leq k \leq K, \tag{138}$$

and there exists a constant c_1, such that the sequence

$$\bar{\mathbf{p}}_k^{(n+1)} = \frac{1}{C(\mathbf{\Gamma}_Q)} \cdot \gamma_k \mathcal{I}_k(\bar{\mathbf{p}}^{(n)}) \tag{139}$$

is lower-bounded by c_1. Then, $c(\mathbf{\Gamma}_Q) = C(\mathbf{\Gamma}_Q)$ and there exists a $\hat{\mathbf{p}} > 0$ such that (135) holds.

Proof. The proof is in analogy to the proof of Theorem 22. One can exploit that $\bar{\mathbf{p}}_k^{(n)}$ is monotonically decreasing in n. □

7. CONCLUSIONS

In this paper, we introduce an analytical framework for SIR balancing, based on an axiomatic interference model. This abstract approach has the advantage that it still holds when considering adaptive receiver designs or other concepts that affect the interference. The only requirement is that the axioms (A1)–(A4) are fulfilled. Known results on SIR balancing, which are based on a *fixed* irreducible coupling matrix, are included as a special case.

The SIR balancing problem completely characterizes the QoS feasible region of a multiuser system. Thus, the results provide a deeper insight into the performance tradeoff between multiple users in an interference-limited system. Since our approach includes power control, optimal receive strategies, and QoS provision, it offers an integral approach to cross-layer optimization.

The first contribution of this work is to characterize the existence of balancing power allocations for general interference functions. For special cases, we can prove additional properties like continuity and uniqueness. In particular, there is always a unique positive power allocation for $K = 2$ and 3 users. This need not hold for $K \geq 4$, which has been demonstrated by examples.

Then, it is shown how the general SIR balancing theory can be connected with matrix theory. It is shown that the max-min-SIR optimum equals the optimum obtained by eigenvalue optimization. Thus, both strategies, which are conceptually different, can equivalently be used to describe the SIR feasible region. There is an interesting link between the SIR balancing theory and known results from the theory of irreducible matrices. But there is no general duality between both problems. The equivalence does not extend to the optimizers. Examples have been given for cases where different behaviours occur.

Additional properties can be shown if the interference is described by a coupling matrix. But even for this more specific model, the general results can be shown to be strict. The matrix approach offers some additional insights, like the connection with the max-min approach and the optimization of the spectral radius.

One big advantage of the axiomatic SIR balancing theory, as compared to the matrix-based model, is that it applies to a larger class of potential problems. Additional requirements and constraints may be easily included in the interference functions. Thus, it can be expected that the theory will be useful for the development of future cross-layer concepts and algorithms.

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