Guo *et al. EURASIP Journal on Wireless Communications and Networking* (2017) 2017:168
DOI 10.1186/s13638-017-0954-8

EURASIP Journal on Wireless Communications and Networking

RESEARCH

Open Access

Multichannel analysis of soft frequency reuse and user association in two-tier heterogeneous cellular networks

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Abstract

The cell range expansion (CRE) is encouraged to be applied in the heterogeneous cellular networks (HCNs), to enhance the capacity by offloading macro users to small cells. However, the enhanced inter-cell interference coordination (eICIC) techniques are supposed to be used for mitigating the strong cross-tier interference suffered by the offloaded users and small cell edge users. To address this, a novel soft frequency reuse (SFR) scheme is adopted in this paper. We analyze multichannel downlink scenarios for the SFR scheme using the tools of stochastic geometry. In consideration of the random resource allocation and practical cell load model, the analytical results of coverage probability and average user rate are derived and validated through Monte Carlo methods. Furthermore, our results can reduce to simple closed-form under reasonable special case for modern urban cellular networks. The main evaluation of the performance in terms of average user rate is presented, and the optimal combination of association bias and parameters of the SFR scheme is also investigated. Numerical results show that the SFR scheme outperforms the frequency resource partitioning (FRP) scheme in any load condition. Moreover, the CRE with SFR scheme can improve the average user rate significantly.

Keywords: Heterogeneous cellular networks, Cell range expansion, Cell load, eICIC, SFR, Stochastic geometry

1 Introduction

Heterogeneous cellular networks (HCNs) are expected to be one of primary technologies for the emerging fifth generation (5G) mobile networks [1, 2], which can be considered as an efficient solution for dealing with the exploding data traffic demands of users. In a HCN, it is proved that the co-channel deployment of low-power base stations (BSs) (also known as small cells such as micro, pico, and femto BSs) with conventional macro BSs can yield the largest sum rate [3].

For a typical network, its user association rule that connects a user to a specific serving BS could substantially affect the network performance. In the existing conventional homogeneous networks, the maximum received signal strength (max-RSS) is widely adopted, which can achieve the anticipated performance. However, the max-RSS user association rule is not suitable for HCNs, since the large difference in transmit powers between small cells (e.g., pico BSs $\approx$ 30 dBm) and macro BSs($\approx$ 46 dBm) [4], and thereby most of users will be associated with the macro BSs. The unbalanced user load leads to overloaded macro BSs and inefficient resource utilization in small cells. To cope with this problem, the biased user association also known as cell range expansion (CRE) has been proposed [5], wherein the macro users are proactively offloaded to small cells. Nevertheless, the drawback of biased user association is that the offloaded macro users referred to as range-expanded small cell users are liable to experience severe interference from the nearby macro BSs. Naturally, the small cell edge users also are vulnerable users. In this context, the enhanced inter-cell interference coordination (eICIC) techniques [6] are expected to mitigate such strong interference.

1.1 Motivation and related work

Previous works on the CRE in conjunction with suitable eICIC techniques can be divided into two general groups, namely, time-domain strategies [7–13] and

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frequency-domain strategies [14–18]. One of time-domain strategies is the so-called almost blanking subframes (ABS) introduced in 3GPP Release 10, where macro BSs are periodically muted in order to mitigate interference to offloaded users [7–10]. The ABS can be considered as a resource partitioning in the time domain. It can be seen in [8] that the user throughput can be enhanced through simulations. Using the tools of stochastic geometry, tractable expressions which can give clear insight into the performance gain are derived in [7, 9]. Furthermore, the work [10] analyzes the required number of ABS based on user throughput requirement, but the CRE is not captured. However, a larger association bias is required in a heavily loaded scenario, which causes macro BSs should be shut off about half the time. This is counter-intuitive and unrealistic. Hence, the reduced power subframes (RPS) are encouraged to be applied to address this issue [11–13]. Instead of ABS, the RPS are allocated to macro interior users with lower transmit power. The capacity loss caused by ABS is thereby reduced because the sever interference to the small cell edge users is mitigated. The results in [13] show that the RPS provide better rate coverage than the ABS, with a given association bias.

We focus on the frequency-domain strategies in this paper. In [14, 15], the authors propose a frequency resource partitioning (FRP) scheme, wherein a certain fraction of frequency resources with full transmit power is preserved for offloaded users. Analytical expressions for coverage probability and rate distribution are presented in [14] using the tools of stochastic geometry. The optimal system performance can be achieved through jointly tune association bias and resource partitioning fraction. However, the full-load model that all the BSs are always active is not reasonable, especially for the network with CRE. This is because the assumption of full-load model cannot adequately reflect the enhancement in performance through load balancing. In detail, the interference from a BS is mainly dependent on its load. That is, the probability of a BS becomes the source of interference is directly proportional to its users. Taking such effects into consideration, the authors of [15, 19] propose a more practical cell load model that a BS is active on a given sub-channel only if the sub-channel is allocated for at least one user. In [15], both the coverage probability and average user rate of FRP scheme are dependent on user density and the resource partitioning fraction, apart from association bias. The results in [15] are more practical for design guidelines.

Nevertheless, the drawback of FRP scheme is the sacrifice of the spectral efficiency of macro users. Hence, the work [16] presents an evolved FRP scheme, wherein a certain fraction of frequency resources is allocated to not only offloaded macro users with full transmit power but also macro users with lower transmit power. By means of the transmit power reduction in macro tier, the spectral efficiency of macro users is guaranteed. However, the transmit power reduction is randomly applied to macro users, which probably leads to that the performance of macro users in poor situation gets worse. In addition, the small cell edge users also are vulnerable to interfered by the nearby macro BSs. Similar to the RPS, a novel soft frequency reuse (SFR) scheme proposed by [17, 18, 20] can mitigate the strong cross-tier interference by means of joint resource partitioning and transmit power reduction for macro interior users. The strong cross-tier interference suffered by offloaded macro users and pico edge users is commonly mitigated. More recently, the work [17] presents the spectral efficiency analysis for the proposed SFR scheme based on stochastic geometry. As a result, the system spectral efficiency can be significantly improved. Although the proportional fair resource allocation is adopted in [17], the analytical results cannot adequately capture the variation in performance because of the assumption of full-load model. To the best of our knowledge, none of the earlier works considered the impact of resource allocation and appropriate load model on the SFR based on stochastic geometry, which is fulfilled in this paper.

1.2 Approach and contributions

Motivated by the works in [15, 19], we propose a general and tractable framework to analyze joint SFR, appropriate load model and CRE in a two-tier HCNs. Based on stochastic geometry, the locations of the BSs in each tier are modeled as a two-dimensional homogeneous Poisson Point Process (HPPP) [21–23], which have been proved to be as accurate as the hexagonal grid model. Furthermore, the Fractional Frequency Reuse (FFR) was analyzed using PPPs in order to tackle the strong interference from nearby macro BSs [24, 25]. In these works of [21–25], all analytical results have been verified via Monte Carlo methods. Note that all these works just adopt a simple resource allocation scheme, namely, the entire bandwidth of each BS is time-shared for its users. We consider a multichannel downlink based on orthogonal frequency division multiple access (OFDMA) technique, where each user is only served by one sub-channel. In this paper, we use the metric of average user rate, which can reflect the high spectral efficiency obtained from the proposed SFR scheme.

Based on the approach described above, the contributions of the paper are summarized as follows:

1) Based on the cell load model, we first derive the coverage probability of the network without SFR scheme, which can be used for the derivation of coverage probability and average rate of the network with SFR scheme.
2) Next, we derive the evaluation for network performance of the proposed SFR scheme is

performed in terms of average user rate incorporated with the random resource allocation and cell load. For special case, the expressions of coverage probability are simple closed-form.
3) We compare the proposed SFR scheme with FRP scheme. Moreover, we comprehensively analyze the average user rate under different parameters by varying the association bias, resource partitioning factor, power control factor, and SINR thresholds. Then, we show the impact of the aforementioned parameters on the average user rate and investigate the optimal combination finally.
4) We show that the proposed SFR scheme is promising for improving the average user rate while considering the cell load.

The remainder of the paper is organized as follows: the system model, user association scheme, cell load analysis, and SFR scheme are presented in Section 2. Mathematical coverage probability and average user rate expressions are derived in Section 3. In Section 4, numerical results on the performance evaluation for the proposed SFR scheme are analyzed. Furthermore, the comparison with FRP scheme is presented. Finally, the paper is concluded in Section 5.

2 System model

2.1 Two-tier cellular network model

We consider a two-tier downlink cellular network based on OFDMA technique, i.e., the intra-cell interference is not considered. Without any loss of generality, let the first tier be macro (higher-power BS) tier, while let the second tier be pico (lower-power BS) tier. The locations of the BSs in jth tier are modeled as a two-dimensional HPPP Φ_j with density λ_j. Furthermore, the users are located according to another HPPP Φ_u with density λ_u, which is independent of $\left\{\Phi_j\right\}_{j=1,2}$. For the co-channel deployment of the network, both the network tiers share the same set of available sub-channels (denoted by $\mathbf{C}$). Then, let $|\mathbf{C}|$ denote the total number of available sub-channels. Moreover, every BS in jth tier transmits with same power P_j^T and thus, the power per sub-channel of a BS in jth tier is kept constant at $P_j = P_j^T \big/ |\mathbf{C}|$. For tractability, the standard power loss propagation model is applied in both tiers with the same path loss exponent $\alpha > 2$. As far as random channel fluctuations, Rayleigh fading with mean 1 (denoted as $H_x \sim \exp(1)$) is applied at each channel. The noise is assumed to be additive with power σ^2. All macro and pico BSs are assumed to be open access in this paper. That is, the number of users served by each BS is unlimited.

2.2 User association

We consider a user association that each user chooses its serving BS based on maximum biased-received-power (BRP) (termed biased user association) [7–9, 15–17]. Denote R_j as the distance of the typical user from its nearest BS of jth tier. A typical user at the origin is associated with the nearest BS in the kth tier if

$$k = \arg\max\left\{P_1 R_1^{-\alpha}, P_2 B R_2^{-\alpha}\right\}, \tag{1}$$

where $B \geq 0$ dB is the association bias for tier 2 (pico tier). For simplicity, it is assumed that the association bias for tier 1 (macro tier) is unity in this paper. As the association bias B increases, more macro users will be offloaded to the corresponding pico BSs. Here, we let $\mathcal{U}_k$ denote the set of users in the kth tier, which satisfies $\mathcal{U}_1 \cup \mathcal{U}_2 = \mathcal{U}$.

Based on (1), the probabilities for the typical user to associate with macro and pico tiers, denoted by $\mathcal{A}_1$ and $\mathcal{A}_2$, can be derived as in [22], i.e.,

$$\mathcal{A}_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2\left(\frac{P_2 B}{P_1}\right)^{2/\alpha}} \text{ and } \mathcal{A}_2 = \frac{\lambda_2}{\lambda_1\left(\frac{\mathrm{P}_1}{\mathrm{P}_2\mathrm{B}}\right)^{2/\alpha} + \lambda_2}. \tag{2}$$

Using (2), the probability density function (PDF) of the distance between the typical user and its serving BS, denoted by $f_{X_k}(x)$, is also given in [22] as

$$f_{X_k}(x) = \frac{2\pi\lambda_k x}{\mathcal{A}_k}\exp\left(\frac{-\pi\lambda_k x^2}{\mathcal{A}_k}\right). \tag{3}$$

Since all BSs in every tier transmit with the same transmit power, each user belongs to macro tier or pico tier is always associated with the nearest macro BS or pico BS based on the maximum BRP. The two-tier cellular network coverage region will be constituted by two independent Voronoi tessellations. Furthermore, the size of each Voronoi cell is an i.i.d. random variable [26]. On the basis of statistical property of the PPP, the probability mass function (PMF) of the number of users associated with a randomly chosen kth tier BS can be expressed as

$$\mathbb{P}(N_k = n) = \frac{3.5^{3.5}}{n!}\frac{\Gamma(n+3.5)}{\Gamma(3.5)}\left(\frac{\lambda_u\mathcal{A}_k}{\lambda_k}\right)^n\left(3.5 + \frac{\lambda_u\mathcal{A}_k}{\lambda_k}\right)^{-(n+3.5)}, \tag{4}$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the standard gamma function. The statistical property of the number of users is crucial for calculating a cell load.

2.3 The main results of network without SFR

Before discussing the proposed SFR scheme, we firstly focus on the network's SINR distribution without SFR, which is extremely essential for the classification of users. Following the analysis [22, 23], the main results for the biased user association are briefly presented here for the purpose of analytical evaluating the proposed SFR scheme.

2.3.1 Resource allocation and load statistics

For the sake of keeping simplicity and tractability of the PPP model, we adopt a simple random resource allocation scheme for both macro and pico users. That is, each BS, independent of the other BSs, randomly and uniformly selects one sub-channel for each of its users. If the number of users in a serving BS is greater than the number of available sub-channels, the resources can be equally allocated in time-division way. Hence, together with the use of OFDMA, the intra-cell interference can be ignored. With regard to other sophisticated resource allocation schemes like opportunistic and fair resource allocation, they are not analytically tractable and will be taken in consideration for our future work.

Different from the assumptions in [14–17] that all BSs of each tier are always active, we utilize a more practical load model that a BS is active when it has at least one user to serve. When a BS has no user to serve, its corresponding sub-channel set $\mathbf{C}$ will be left idle. Furthermore, it is also assumed that each BS has full buffer traffic downlink transmission for its each user. Hence, a BS is the source of interference when the BS must be active and simultaneously use the same sub-channel.

To obtain the SINR statistics on a given sub-channel, [15, 19] have derived the probability that a typical BS of each tier accesses a given sub-channel, which depends on the PMF of the number of users associated with that BS.

Lemma 1 *Let ρ_k denote the probability that a typical BS in the kth tier accesses a given sub-channel from the set $\mathbf{C}$. Then,*

$$\rho_k = 1 - \sum_{n=0}^{|\mathbf{C}|-1} \left(1 - \frac{n}{|\mathbf{C}|}\right) \mathbb{P}\left(N_k = n\right). \tag{5}$$

Note that, we can term ρ_k as the load of a typical BS in the kth tier. It also can be viewed as the probability that a typical BS becomes the source of interference because the BSs use the same sub-channel. Therefore, the macro BSs and pico BSs using the same sub-channel will form two independent homogeneous PPPs Ψ_1 of density $\lambda_1\rho_1$ and Ψ_2 of density $\lambda_2\rho_2$, respectively. In other words, the interfering sets Ψ_1 and Ψ_2 are independent thinning of the original PPPs Φ_1 and Φ_2, respectively, with retention probabilities ρ_1 and ρ_2, respectively [15, 19].

2.3.2 SINR distribution

The downlink received SINR at the typical user is expressed as

$$\mathrm{SINR}_k = \frac{P_k H_x x^{-\alpha}}{\sum_{j=1}^{2} I_{x,j} + \sigma^2}, \tag{6}$$

where $I_{x,j} = P_j \sum_{y\in\Psi_j\backslash b_k} H_y \left\|y\right\|^{-\alpha}$ is the cumulative interference from all the loaded BSs in the jth tier (except the user's serving BS in the kth tier), and H_x is the channel fading gain from the serving BS b_k at a distance x.

Hence, the SINR distribution of a typical user can be thought of equivalently as the probability that the typical user can achieve a target SINR threshold T, i.e., the conditional coverage probability. The conditional coverage probability when a typical user $u \in \mathcal{U}_k$ is defined as

$$p_{c,k}\left(T\right) \triangleq \mathbb{P}\left(\mathrm{SINR} > T \,| u \in \mathcal{U}_k\right). \tag{7}$$

Theorem 1 *Following the law of total probability, the coverage probability without SFR of a typical user $u \in \mathcal{U}$ is*

$$p_c\left(T\right) = \mathcal{A}_1 p_{c,1}\left(T\right) + \mathcal{A}_2 p_{c,2}\left(T\right), \tag{8}$$

where the conditional coverage probabilities without SFR are given by (9) *and* (10), $Q(a,b,c,d) = c^{2/b} + a^{2/b} d \int_{\left(\frac{c}{a}\right)^{2/b}}^{\infty} \frac{du}{1+u^{b/2}}$, *and* $\mathrm{SNR}_k(x) = \frac{P_k x^{-\alpha}}{\sigma^2}$.

$$\begin{aligned} P_{c,1}(T) = \frac{2\pi\lambda_1}{\mathcal{A}_1}\int_0^{\infty} x \exp\left(-\frac{T}{\mathrm{SNR}_1(x)}\right) \\ \exp\big\{-\pi\lambda_1 Q(T,\alpha,1,\rho_1)x^2 \\ -\pi\lambda_2 (P_2/P_1)^{2/\alpha} Q(T,\alpha,B,\rho_2)x^2\big\}\, dx \end{aligned} \tag{9}$$

$$\begin{aligned} P_{c,2}(T) = \frac{2\pi\lambda_2}{\mathcal{A}_2}\int_0^{\infty} x \exp\left(-\frac{T}{\mathrm{SNR}_2(x)}\right) \\ \exp\big\{-\pi\lambda_1 (P_1/P_2)^{2/\alpha} Q(T,\alpha,1/B,\rho_1)x^2 \\ -\pi\lambda_2 Q(T,\alpha,1,\rho_2)x^2\big\}\, dx \end{aligned} \tag{10}$$

Proof The proof is given in Appendix A. □

Unlike the results in [22], the coverage probability is dependent of ρ_k, i.e., the load of each BS. Additionally, the conditional coverage probabilities without SFR are of special importance with respect to the proposed SFR scheme.

2.4 SFR scheme

In this paper, the SFR scheme proposed by [17, 18, 20] is applied to our two-tier cellular network, as shown in Fig. 1.

2.4.1 Resource partitioning and power control

Similarly, both the network tiers are co-channel deployment. However for each cell, the entire available sub-channel set $\mathbf{C}$ is divided into two different subsets $\mathbf{C}_1$ and $\mathbf{C}_2$ with size $|\mathbf{C}_1|$ and $|\mathbf{C}_2|$, respectively. The two different sunsets $\mathbf{C}_1$ and $\mathbf{C}_2$ have no intersection, i.e., $\mathbf{C}_1 \cap \mathbf{C}_2 = \emptyset$ and $\mathbf{C} = \mathbf{C}_1 \cup \mathbf{C}_2$. Therefore, a resource partitioning factor η is defined as $\eta = |\mathbf{C}_1|/|\mathbf{C}|$ $(0 \le \eta \le 1)$.

In the proposed SFR scheme, a typical BS classifies users with average SINR as two types of users: interior users and edge users. Instead of a geographic classification criterion, the SINR threshold can more adequately

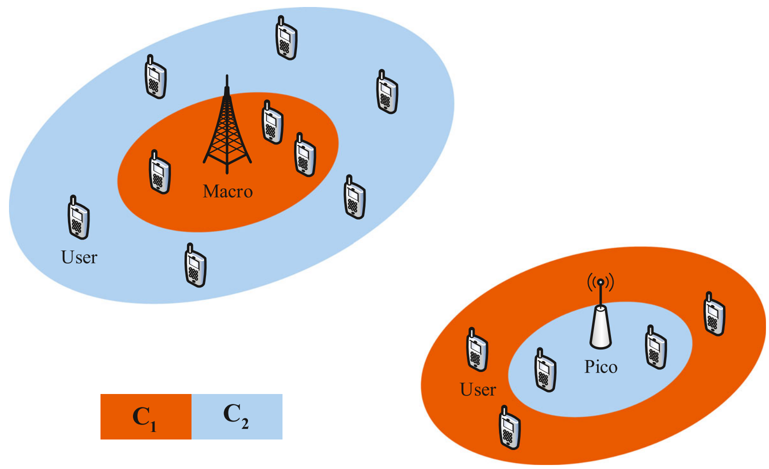

Fig. 1 Illustration of SFR scheme

capture the randomness: the locations of the BSs and users [24, 25]. Let $T_{\mathrm{FR},k}$ denote the SFR threshold of a typical BS in the kth tier. With SFR, each user calculates its SINR to the serving BS in the kth tier, and if it is less than the threshold $T_{\mathrm{FR},k}$, then the user is an edge user, and otherwise the user is an interior user.

In this paper, we consider the following random resource allocation scheme. Let η be the fraction of resources (namely, $\mathbf{C}_1$) allocated to the macro interior users and pico edge users. The remaining $1-\eta$ fraction of resources (namely, $\mathbf{C}_2$) are allocated to the macro edge users and pico interior users. Nevertheless, the macro edge users and pico edge users, especially the offloaded macro users, usually suffer severe interference from the neighboring macro BSs. To accomplish this, a power control factor β $(0<\beta<1)$ is introduced to the sub-channel set $\mathbf{C}_1$ used by the macro BSs, i.e., $P_1^{\mathrm{i}}=\beta P_1$ and $P_1^{\mathrm{e}}=P_1$, where P_1^{i} is the transmit power of macro BSs for the interior users and P_1^{e} is the transmit power of macro BSs for the edge users. For all the pico users, the transmit power keeps full power transmission, the same as the macro edge users. Therefore, both high spectral efficiency and good user experience of the pico edge users can be achieved.

2.4.2 User association and load statistics

According to the proposed SFR scheme and user association scheme above, a typical user $u\in\mathcal{U}$ can lie in the following four disjoint sets:

$$u\in\begin{cases}\mathcal{U}_1^{\mathrm{i}}, \text{if } k=1, P_1R_1^{-\alpha}\geq P_2BR_2^{-\alpha} \ \&\ \mathrm{SINR}_1\geq T_{\mathrm{FR},1}\\ \mathcal{U}_1^{\mathrm{e}}, \text{if } k=1, P_1R_1^{-\alpha}\geq P_2BR_2^{-\alpha} \ \&\ \mathrm{SINR}_1< T_{\mathrm{FR},1}\\ \mathcal{U}_2^{\mathrm{i}}, \text{if } k=2, P_1R_1^{-\alpha}< P_2BR_2^{-\alpha} \ \&\ \mathrm{SINR}_2\geq T_{\mathrm{FR},2}\\ \mathcal{U}_2^{\mathrm{e}}, \text{if } k=2, P_1R_1^{-\alpha}< P_2BR_2^{-\alpha} \ \&\ \mathrm{SINR}_2< T_{\mathrm{FR},2}\end{cases}, \quad (11)$$

where $\mathcal{U}_1^{\mathrm{i}}$ is the set of interior users associated with the kth tier BSs and $\mathcal{U}_1^{\mathrm{e}}$ is the set of edge users associated with the kth tier BSs. Clearly, $\mathcal{U}_1\triangleq\mathcal{U}_1^{\mathrm{i}}\cup\mathcal{U}_1^{\mathrm{e}}$ is the set of macro users, $\mathcal{U}_2\triangleq\mathcal{U}_2^{\mathrm{i}}\cup\mathcal{U}_2^{\mathrm{e}}$ is the set of pico users, and $\mathcal{U}_1^{\mathrm{i}}\cup\mathcal{U}_1^{\mathrm{e}}\cup\mathcal{U}_2^{\mathrm{i}}\cup\mathcal{U}_2^{\mathrm{e}}=\mathcal{U}$.

For a randomly chosen user in Φ_u, it will exactly belong to specific one of above four sets, according to the user association strategy in (11). Then, the probabilities that a randomly chosen user belongs to the sets $\mathcal{U}_k^{\mathrm{i}}$ and $\mathcal{U}_k^{\mathrm{e}}$ are $\mathcal{A}_kp_{c,k}\left(T_{\mathrm{FR},k}\right)$ and $\mathcal{A}_k\left(1-p_{c,k}\left(T_{\mathrm{FR},k}\right)\right)$, which also can be interpreted as the average fraction of users belonging to the sets $\mathcal{U}_k^{\mathrm{i}}$ and $\mathcal{U}_k^{\mathrm{e}}$. Irrespective of the exact distribution of user locations, the numbers of interior and edge users in a typical BS are significant for characterizing the cell load. To make the proposed framework analytically tractable, the sets $\mathcal{U}_k^{\mathrm{i}}$ and $\mathcal{U}_k^{\mathrm{e}}$ can be equivalently modeled as independent homogeneous PPPs with densities $\mathcal{A}_kp_{c,k}\left(T_{\mathrm{FR},k}\right)$ and $\mathcal{A}_k\left(1-p_{c,k}\left(T_{\mathrm{FR},k}\right)\right)$, respectively [15]. Based on the same lines as the derivation in (4), their PMFs are given in the following lemma.

Lemma 2 *Let N_k^{i} and N_k^{e} denote the numbers of interior and edge users associated with a BS in the kth tier, respectively. Their PMFs are given by* (12) *and* (13).

$$\mathbb{P}\left(N_k^{\mathrm{i}}=n\right)=\frac{3.5^{3.5}}{n!}\frac{\Gamma\left(n+3.5\right)}{\Gamma\left(3.5\right)}\left(\frac{\lambda_u\mathcal{A}_kp_{c,k}\left(T_{\mathrm{FR},k}\right)}{\lambda_k}\right)^n \left(3.5+\frac{\lambda_u\mathcal{A}_kp_{c,k}\left(T_{\mathrm{FR},k}\right)}{\lambda_k}\right)^{-(n+3.5)} \quad (12)$$

$$\mathbb{P}\left(N_k^{\mathrm{e}}=n\right)=\frac{3.5^{3.5}}{n!}\frac{\Gamma\left(n+3.5\right)}{\Gamma\left(3.5\right)}\left(\frac{\lambda_u\mathcal{A}_k\left(1-p_{c,k}\left(T_{\mathrm{FR},k}\right)\right)}{\lambda_k}\right)^n \left(3.5+\frac{\lambda_u\mathcal{A}_k\left(1-p_{c,k}\left(T_{\mathrm{FR},k}\right)\right)}{\lambda_k}\right)^{-(n+3.5)} \quad (13)$$

Lemma 3 *Let ρ_k^{i} and ρ_k^{e} denote the probabilities that a typical BS in the kth tier accesses a given sub-channel from the allocated subsets $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively. Then,*

$$\rho_1^{\mathrm{i}} = 1 - \sum_{n=0}^{|\mathbf{C}_1|-1} \left(1 - \frac{n}{|\mathbf{C}_1|}\right) \mathbb{P}\left(N_1^{\mathrm{i}} = n\right), \tag{14}$$

$$\rho_1^{\mathrm{e}} = 1 - \sum_{n=0}^{|\mathbf{C}_2|-1} \left(1 - \frac{n}{|\mathbf{C}_2|}\right) \mathbb{P}\left(N_1^{\mathrm{e}} = n\right), \tag{15}$$

$$\rho_2^{\mathrm{i}} = 1 - \sum_{n=0}^{|\mathbf{C}_2|-1} \left(1 - \frac{n}{|\mathbf{C}_2|}\right) \mathbb{P}\left(N_2^{\mathrm{i}} = n\right), \tag{16}$$

$$\rho_2^{\mathrm{e}} = 1 - \sum_{n=0}^{|\mathbf{C}_1|-1} \left(1 - \frac{n}{|\mathbf{C}_1|}\right) \mathbb{P}\left(N_2^{\mathrm{e}} = n\right). \tag{17}$$

As our explanation earlier, we can term ρ_k^{i} and ρ_k^{e} as the interior load and edge load of a typical BS in the kth tier, respectively. For a macro interior user served on a sub-channel from the subset $\mathbf{C}_1$, it will be interfered by the interior loaded macro BSs and the edge loaded pico BSs. Therefore, its interfering sets Ξ_1^{i} and Ξ_2^{e} are independent thinning of the original PPPs Φ_1 and Φ_2, respectively, with retention probabilities ρ_1^{i} and ρ_2^{e}, respectively. Similarly, the interfering sets Ξ_k^{i} and Ξ_k^{e} are two independent homogeneous PPPs with densities $\lambda_k \rho_k^{\mathrm{i}}$ and $\lambda_k \rho_k^{\mathrm{e}}$, respectively. The interfering sets of each user can be obtained as given in Table 1.

By employing the user association scheme and SFR scheme described above, the downlink received SINR of a typical user $u \in \mathcal{U}$ located at the origin is given by

$$\begin{aligned}\mathrm{SINR} = &\mathbf{1}\left(u \in \mathcal{U}_1^{\mathrm{i}}\right) \frac{\beta P_1 H_x x^{-\alpha}}{\beta I_{x,1}^{\mathrm{i}} + I_{x,2}^{\mathrm{e}} + \sigma^2} + \mathbf{1}\left(u \in \mathcal{U}_1^{\mathrm{e}}\right) \frac{P_1 H_x x^{-\alpha}}{I_{x,1}^{\mathrm{e}} + I_{x,2}^{\mathrm{i}} + \sigma^2} \\ &+ \mathbf{1}\left(u \in \mathcal{U}_2^{\mathrm{i}}\right) \frac{P_2 H_x x^{-\alpha}}{I_{x,1}^{\mathrm{e}} + I_{x,2}^{\mathrm{i}} + \sigma^2} + \mathbf{1}\left(u \in \mathcal{U}_2^{\mathrm{e}}\right) \frac{P_2 H_x x^{-\alpha}}{\beta I_{x,1}^{\mathrm{i}} + I_{x,2}^{\mathrm{e}} + \sigma^2},\end{aligned} \tag{18}$$

where $\mathbf{1}(A)$ is the indicator function that takes the value 1 if the event A is true, H_x is the channel fading gain from the serving BS b_k at a distance x, and $I_{x,j}^{\mathrm{i}} = P_j \sum_{y \in \Xi_j^{\mathrm{i}} \setminus b_k} H_y \|y\|^{-\alpha}$ and $I_{x,j}^{\mathrm{e}} = P_j \sum_{y \in \Xi_j^{\mathrm{e}} \setminus b_k} H_y \|y\|^{-\alpha}$ are the cumulative interference from the interior and edge loaded BSs in the jth tier, respectively.

Table 1 The Interfering sets of a typical user

User type	Interfering sets
Macro interior user ($u \in \mathcal{U}_1^{\mathrm{i}}$)	$\Xi_1^{\mathrm{i}}, \Xi_2^{\mathrm{e}}$
Macro edge user ($u \in \mathcal{U}_1^{\mathrm{e}}$)	$\Xi_1^{\mathrm{e}}, \Xi_2^{\mathrm{i}}$
Pico interior user ($u \in \mathcal{U}_2^{\mathrm{i}}$)	$\Xi_1^{\mathrm{e}}, \Xi_2^{\mathrm{i}}$
Pico edge user ($u \in \mathcal{U}_2^{\mathrm{e}}$)	$\Xi_1^{\mathrm{i}}, \Xi_2^{\mathrm{e}}$

3 Average user rate

This section is our main technical part. We first derive the general coverage probability for the proposed SFR scheme. Then, the methods of derivation are subsequently used for the average user rate. Moreover, we present a special case where $\alpha = 4$ and $\sigma^2 = 0$, representing for the modern cellular networks. For this case, the expressions of coverage probability reduce to simple closed-form, which can provide clear insight into the performance analysis of each user.

3.1 Coverage probability

In the context of this paper, the conditional coverage probabilities $p_{\mathrm{FFR},k}^{\mathrm{i}}(T)$ and $p_{\mathrm{FFR},k}^{\mathrm{e}}(T)$ can be defined, respectively, as

$$p_{\mathrm{FFR},k}^{\mathrm{i}}(T) \triangleq \mathbb{P}\left(\mathrm{SINR} > T \,\middle|\, u \in \mathcal{U}_k^{\mathrm{i}}\right), \tag{19}$$

$$p_{\mathrm{FFR},k}^{\mathrm{e}}(T) \triangleq \mathbb{P}\left(\mathrm{SINR} > T \,\middle|\, u \in \mathcal{U}_k^{\mathrm{e}}\right). \tag{20}$$

Following from the law of total probability, the coverage probability of a typical user following the user association strategy in (11) is given by (21).

$$\begin{aligned}p_{c,\mathrm{FFR}}(T) = \sum_{k=1}^{2} \mathcal{A}_k &\left(p_{c,k}\left(T_{\mathrm{FR},k}\right) p_{\mathrm{FFR},k}^{\mathrm{i}}(T)\right. \\ &\left.+ \left(1 - p_{c,k}\left(T_{\mathrm{FR},k}\right)\right) p_{\mathrm{FFR},k}^{\mathrm{e}}(T)\right)\end{aligned} \tag{21}$$

$$p_{\mathrm{FFR},1}^{\mathrm{i}}(T) = \frac{\begin{aligned}\frac{2\pi\lambda 1}{\mathcal{A}1} \int_0^\infty x \exp\left(-\frac{T}{\mathrm{SNR}_1(x,\beta)} - \frac{T_{\mathrm{FR},1}}{\mathrm{SNR}_1(x,1)}\right) \exp\Big\{-\pi\lambda_1 x^2 \left[1 + 2\xi\left(\rho_1^{\mathrm{i}}, \rho_1, T, T_{\mathrm{FR},1}, \alpha\right)\right] \\ -\pi\lambda_2 \left(\frac{P_2 B}{P_1}\right)^{\frac{2}{\alpha}} x^2 \left[1 + 2\xi\left(\rho_2^{\mathrm{e}}, \rho_2, (\beta B)^{-1} T, B^{-1} T_{\mathrm{FR},1}, \alpha\right)\right]\Big\}\, dx\end{aligned}}{p_{c,1}(T_{\mathrm{FR},1})} \tag{22}$$

$$p_{\mathrm{FFR},1}^{\mathrm{i}}(T) = \frac{\lambda_1\left(1 + \rho_1\sqrt{T_{\mathrm{FR},1}} \arctan(\sqrt{T_{\mathrm{FR},1}})\right) + \lambda_2\sqrt{P_2/P_1}\left(\sqrt{B} + \rho_2\sqrt{T_{\mathrm{FR},1}} \arctan\left(\sqrt{T_{\mathrm{FR},1}/B}\right)\right)}{\lambda_1\left(1 + 2\xi\left(\rho_1^{\mathrm{i}}, \rho_1, T, T_{\mathrm{FR},1}, 4\right)\right) + \lambda_2\sqrt{P_2 B/P_1}\left(1 + 2\xi\left(\rho_2^{\mathrm{e}}, \rho_2, (\beta B)^{-1} T, B^{-1} T_{\mathrm{FR},1}, 4\right)\right)} \tag{23}$$

$$\begin{aligned}p_{\mathrm{FFR},1}^{\mathrm{e}}(T) = &\frac{\frac{2\pi\lambda_1}{\mathcal{A}_1} \int_0^\infty x \exp\left(-\frac{T}{\mathrm{SNR}_1(x,1)}\right) \exp\left\{-\pi\lambda_1 Q\left(T, \alpha, 1, \rho_1^{\mathrm{e}}\right) x^2 - \pi\lambda_2 \left(\frac{P_2}{P_1}\right)^{\frac{2}{\alpha}} Q\left(T, \alpha, B, \rho_2^{i}\right) x^2\right\} dx}{1 - p_{c,1}(T_{\mathrm{FR},1})} \\ &- \frac{\begin{aligned}\frac{2\pi\lambda 1}{\mathcal{A}_1} \int_0^\infty x \exp\left(-\frac{T + T_{\mathrm{FR},1}}{\mathrm{SNR}_1(x,1)}\right) \exp\Big\{-\pi\lambda_1 x^2 \left[1 + 2\xi\left(\rho_1^{\mathrm{e}}, \rho_1, T, T_{\mathrm{FR},1}, \alpha\right)\right] \\ -\pi\lambda_2 \left(\frac{P_2 B}{P_1}\right)^{\frac{2}{\alpha}} x^2 \left[1 + 2\xi\left(\rho_2^{\mathrm{i}}, \rho_2, B^{-1} T, B^{-1} T_{\mathrm{FR},1}, \alpha\right)\right]\Big\}\, dx\end{aligned}}{1 - p_{c,1}(T_{\mathrm{FR},1})}\end{aligned} \tag{24}$$

$$p_{\mathrm{FFR},1}^{\mathrm{e}}(T) = \frac{(\lambda_1+\lambda_2\sqrt{P_2B/P_1})\left[\lambda_1\left(1+\rho_1\sqrt{T_{\mathrm{FR},1}}\ \arctan\left(\sqrt{T_{\mathrm{FR},1}}\right)\right)+\lambda_2\sqrt{P_2/P_1}\left(\sqrt{B}+\rho_2\sqrt{T_{\mathrm{FR},1}}\ \arctan\left(\sqrt{T_{\mathrm{FR},1}/B}\right)\right)\right]}{(1-\lambda_1-\lambda_2\sqrt{P_2B/P_1})\left[\lambda_1\left(1+\rho_1^{\mathrm{e}}\sqrt{T}\ \arctan\ (\sqrt{T})\right)+\lambda_2\sqrt{P_2/P_1}\left(\sqrt{B}+\rho_2^{\mathrm{i}}\sqrt{T}\ \arctan(\sqrt{T/B})\right)\right]} - \frac{(\lambda_1+\lambda_2\sqrt{P_2B/P_1})\left[\lambda_1\left(1+\rho_1\sqrt{T_{\mathrm{FR},1}}\ \arctan\left(\sqrt{T_{\mathrm{FR},1}}\right)\right)+\lambda_2\sqrt{P_2/P_1}\left(\sqrt{B}+\rho_2\sqrt{T_{\mathrm{FR},1}}\ \arctan\left(\sqrt{T_{\mathrm{FR},1}/B}\right)\right)\right]}{(1-\lambda_1-\lambda_2\sqrt{P_2B/P_1})[\lambda_1(1+2\xi(\rho_1^{\mathrm{e}},\rho_1,T,T_{\mathrm{FR},1},4))+\lambda_2\sqrt{P_2B/P_1}(1+2\xi(\rho_2^{i},\rho_2,B^{-1}T,B^{-1}T_{\mathrm{FR},1},4))]} \quad (25)$$

$$p_{\mathrm{FFR},2}^{\mathrm{i}}(T) = \frac{\frac{2\pi\lambda_2}{\mathcal{A}_2}\int_0^\infty x\exp\left(-\frac{T+T_{\mathrm{FR},2}}{\mathrm{SNR}_2(x,1)}\right)\exp\left\{-\pi\lambda_2x^2\left[1+2\xi\left(\rho_2^{\mathrm{i}},\rho_2,T,T_{\mathrm{FR},2},\alpha\right)\right] -\pi\lambda_1\left(\frac{P_1}{P_2B}\right)^{\frac{2}{\alpha}}x^2\left[1+2\xi\left(\rho_1^{\mathrm{e}},\rho_1,BT,BT_{\mathrm{FR},2},\alpha\right)\right]\right\}dx}{p_{c,2}\left(T_{\mathrm{FR},2}\right)} \quad (26)$$

As discussed in Section 2, we observe that the conditional coverage probability of a typical user depends on two SINR thresholds [24, 25], first the SINR threshold (i.e., $T_{\mathrm{FR},k}$) on the allocated a sub-channel from the entire available sub-channel set $\mathbf{C}$ to determine its status (interior or not), and second the actual SINR threshold (i.e., T) on the newly allocated a sub-channel from the subset $\mathbf{C}_1$ or $\mathbf{C}_2$ to determine whether it is covered or not. Actually, these two SINR thresholds are correlated because the interference may be generated by the same set of BSs, which makes our analysis challenging. The following theorems give the conditional coverage probabilities for a typical user under different status.

Theorem 2 *(Macro tier, interior user): the coverage probability of the macro interior user is given by* (22)*, where* $\mathrm{SNR}_k(x,a) = \frac{aP_kx^{-\alpha}}{\sigma^2}$*,* $p_{c,1}\left(T_{\mathrm{FR},1}\right)$ *is given by* (9)*, and* $\xi(a,b,c,d,e) = \int_1^\infty\left[1-\left(1-a\left(1-\frac{1}{1+cv^{-e}}\right)\right)\left(1-b\left(1-\frac{1}{1+dv^{-e}}\right)\right)\right]vdv$.

Proof The proof is given in Appendix B. □

As we can see, $\xi(a,b,c,d,e)$ is similar to $\rho_jZ(a,b,c)$ given by previous results in Theorem 1, they are different because of the dependence of two SINR thresholds. Moreover, $\xi(a,b,c,d,e)$ can efficiently capture the disparity of the intra-tier and inter-tier interference before and after the proposed SFR scheme is applied.

Now, we turn our attention to the special case where $\alpha = 4$ and $\sigma^2 = 0$, which is significant in practice that is widely applied in lots of literatures [21–25]. It is noted that the typical HCNs are interference-limited which the noise can be ignored compared to the interference. Furthermore, the special case can be considered as the scenario corresponding to an interference-limited urban cellular network [24, 25, 27], where FFR has been generally applied. In the special case, the expression (22) will be further simplified to a simple closed-form.

Corollary 1 *For* $\alpha = 4$ *and* $\sigma^2 = 0$*, the coverage probability of the macro interior user is given by* (23)*, where* $\xi\left(\rho_1^{\mathrm{i}},\rho_1,T,T_{\mathrm{FR},1},4\right) =$

$$\frac{\rho_1^{\mathrm{i}}\sqrt{T}\arctan\left(\sqrt{T}\right)(T_{\mathrm{FR},1}-T-\rho_1T_{\mathrm{FR},1})}{2(T_{\mathrm{FR},1}-T)} + \frac{\rho_1\sqrt{T_{\mathrm{FR},1}}\arctan\left(\sqrt{T_{\mathrm{FR},1}}\right)(T_{\mathrm{FR},1}-T+\rho_1^{\mathrm{i}}T)}{2(T_{\mathrm{FR},1}-T)},$$

and $\xi\left(\rho_2^{\mathrm{e}},\rho_2,(\beta B)^{-1}T,B^{-1}T_{\mathrm{FR},1},4\right) =$

$$\frac{\rho_2^{\mathrm{e}}\sqrt{(\beta B)^{-1}T}\arctan\left(\sqrt{(\beta B)^{-1}T}\right)(T_{\mathrm{FR},1}-\beta^{-1}T-\rho_2T_{\mathrm{FR},1})}{2(T_{\mathrm{FR},1}-\beta^{-1}T)} + \frac{\rho_2\sqrt{B^{-1}T_{\mathrm{FR},1}}\arctan\left(\sqrt{B^{-1}T_{\mathrm{FR},1}}\right)(T_{\mathrm{FR},1}-\beta^{-1}T+\rho_2^{\mathrm{e}}\beta^{-1}T)}{2(T_{\mathrm{FR},1}-\beta^{-1}T)}.$$

Proof When $\alpha = 4$, we have

$$Q(a,4,c,d) = \sqrt{c}+\sqrt{ad}\arctan\left(\sqrt{a/c}\right).$$

Then, $\xi(a,b,c,d,4) =$

$$\int_1^\infty\left[1-\left(1-a\left(1-\frac{1}{1+cv^{-4}}\right)\right)\left(1-b\left(1-\frac{1}{1+dv^{-4}}\right)\right)\right]vdv = \frac{a\sqrt{c}\arctan(\sqrt{c})(d-c-bd)+b\sqrt{d}\arctan\left(\sqrt{d}\right)(d-c+ac)}{2(d-c)}.$$

□

Combining with (22) gives the desired result.

We see that the coverage probability of a macro interior user is a function of the SINR threshold T, the SFR threshold $T_{\mathrm{FR},1}$, the power control factor β, the association bias B, and the load ρ_1, ρ_1^{i}, ρ_2 and ρ_2^{e}. Furthermore, the expression will have an indeterminate form when $T = T_{\mathrm{FR},1}$. Let limit $T \to T_{\mathrm{FR},1}$, the expressions will be simplified to $\xi\left(\rho_1^{\mathrm{i}},\rho_1,T,T_{\mathrm{FR},1},4\right) =$

$$\frac{\left(2\rho_1+2\rho_1^{\mathrm{i}}-\rho_1\rho_1^{\mathrm{i}}\right)\sqrt{T_{\mathrm{FR},1}}\arctan\left(\sqrt{T_{\mathrm{FR},1}}\right)}{4} + \frac{\rho_1\rho_1^{\mathrm{i}}T_{\mathrm{FR},1}}{4\left(T_{\mathrm{FR},1}+1\right)},$$

and $\xi\left(\rho_2^{\mathrm{e}},\rho_2,(\beta B)^{-1}T,B^{-1}T_{\mathrm{FR},1},4\right) =$

$$\frac{\rho_2^{\mathrm{e}}(1-\beta^{-1}-\rho_2)\sqrt{(\beta B)^{-1}T_{\mathrm{FR},1}}\arctan\left(\sqrt{(\beta B)^{-1}T_{\mathrm{FR},1}}\right)}{2(1-\beta^{-1})} + \frac{\rho_2\left(1-\beta^{-1}+\rho_2^{\mathrm{e}\beta^{-1}}\right)\sqrt{B^{-1}T_{\mathrm{FR},1}}\arctan\left(\sqrt{B^{-1}T_{\mathrm{FR},1}}\right)}{2(1-\beta^{-1})}.$$

Theorem 3 *(Macro tier, edge user): the coverage probability of the macro edge user is given by* (24).

Proof The proof is given in Appendix C. □

Corollary 2 *For* $\alpha = 4$ *and* $\sigma^2 = 0$*, the coverage probability of the macro edge user is given by* (25)*, where* $\xi\left(\rho_1^{\mathrm{e}}, \rho_1, T, T_{\mathrm{FR},1}, 4\right) =$

$$\frac{\rho_1^{\mathrm{e}}\sqrt{T}\arctan\left(\sqrt{T}\right)\left(T_{\mathrm{FR},1} - T - \rho_1 T_{\mathrm{FR},1}\right)}{2\left(T_{\mathrm{FR},1} - T\right)} + \frac{\rho_1\sqrt{T_{\mathrm{FR},1}}\arctan\left(\sqrt{T_{\mathrm{FR},1}}\right)\left(T_{\mathrm{FR},1} - T + \rho_1^{\mathrm{e}} T\right)}{2\left(T_{\mathrm{FR},1} - T\right)},$$

and $\xi\left(\rho_2^{\mathrm{i}}, \rho_2, B^{-1}T, B^{-1}T_{\mathrm{FR},1}, 4\right) =$

$$\frac{\rho_2^{\mathrm{i}}\sqrt{B^{-1}T}\arctan\left(\sqrt{B^{-1}T}\right)\left(T_{\mathrm{FR},1} - T - \rho_2 T_{\mathrm{FR},1}\right)}{2\left(T_{\mathrm{FR},1} - T\right)} + \frac{\rho_2\sqrt{B^{-1}T_{\mathrm{FR},1}}\arctan\left(\sqrt{B^{-1}T_{\mathrm{FR},1}}\right)\left(T_{\mathrm{FR},1} - T + \rho_2^{\mathrm{i}} T\right)}{2\left(T_{\mathrm{FR},1} - T\right)}.$$

When $T = T_{\mathrm{FR},1}$*, the limit* $T \to T_{\mathrm{FR},1}$ *will simplify as* $\xi\left(\rho_1^{\mathrm{e}}, \rho_1, T, T_{\mathrm{FR},1}, 4\right) =$

$$\frac{\left(2\rho_1 + 2\rho_1^{\mathrm{e}} - \rho_1\rho_1^{\mathrm{e}}\right)\sqrt{T_{\mathrm{FR},1}}\arctan\left(\sqrt{T_{\mathrm{FR},1}}\right)}{4} + \frac{\rho_1\rho_1^{\mathrm{e}T_{\mathrm{FR},1}}}{4\left(T_{\mathrm{FR},1} + 1\right)},$$

and $\xi\left(\rho_2^{\mathrm{i}}, \rho_2, B^{-1}T, B^{-1}T_{\mathrm{FR},1}, 4\right) =$

$$\frac{\left(2\rho_2 + 2\rho_2^{\mathrm{i}} - \rho_2\rho_2^{\mathrm{i}}\right)\sqrt{B^{-1}T_{\mathrm{FR},1}}\arctan\left(\sqrt{B^{-1}T_{\mathrm{FR},1}}\right)}{4} + \frac{\rho_2\rho_2^{\mathrm{i}}B^{-1}T_{\mathrm{FR},1}}{4\left(B^{-1}T_{\mathrm{FR},1} + 1\right)}.$$

Theorem 4 *(Pico tier, interior user): the coverage probability of the pico interior user is given by* (26).

Proof A user $u \in \mathcal{U}_2$ with SINR $> T_{\mathrm{FR},2}$ is allocated a sub-channel from available sub-channel subset $\mathbf{C}_2$. Conditioning on its previous SINR and applying the Bayes' rule, we have $p_{\mathrm{FFR},2}^{\mathrm{i}}(T)$

$$= \mathbb{P}\left(\frac{P_2\hat{H}_x x^{-\alpha}}{\hat{I}_{x,1}^{\mathrm{e}} + \hat{I}_{x,2}^{\mathrm{i}} + \sigma^2} > T \,\middle|\, \frac{P_2 H_x x^{-\alpha}}{I_{x,1} + I_{x,2} + \sigma^2} > T_{\mathrm{FR},2}\right) = \frac{\mathbb{P}\left(\frac{P_2\hat{H}_x x^{-\alpha}}{\hat{I}_{x,1}^{\mathrm{e}} + \hat{I}_{x,2}^{\mathrm{i}} + \sigma^2} > T, \frac{P_2 H_x x^{-\alpha}}{I_{x,1} + I_{x,2} + \sigma^2} > T_{\mathrm{FR},2}\right)}{\mathbb{P}\left(\frac{P_2 H_x x^{-\alpha}}{I_{x,1} + I_{x,2} + \sigma^2} > T_{\mathrm{FR},2}\right)}$$

Following the method of Theorem 2 gives the desired result. □

Corollary 3 *For* $\alpha = 4$ *and* $\sigma^2 = 0$*, the coverage probability of the pico interior user is given by* (27)*, where* $\xi\left(\rho_2^{\mathrm{i}}, \rho_2, T, T_{\mathrm{FR},2}, 4\right) =$

$$\frac{\rho_2^{\mathrm{i}}\sqrt{T}\arctan\left(\sqrt{T}\right)\left(T_{\mathrm{FR},2} - T - \rho_2 T_{\mathrm{FR},2}\right)}{2\left(T_{\mathrm{FR},2} - T\right)} + \frac{\rho_2\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{T_{\mathrm{FR},2}}\right)\left(T_{\mathrm{FR},2} - T + \rho_2^{\mathrm{i}} T\right)}{2\left(T_{\mathrm{FR},2} - T\right)},$$

and $\xi\left(\rho_1^{\mathrm{e}}, \rho_1, BT, BT_{\mathrm{FR},2}, 4\right) =$

$$\frac{\rho_1^{\mathrm{e}}\sqrt{BT}\arctan\left(\sqrt{BT}\right)\left(T_{\mathrm{FR},2} - T - \rho_1 T_{\mathrm{FR},2}\right)}{2\left(T_{\mathrm{FR},2} - T\right)} + \frac{\rho_1\sqrt{BT_{\mathrm{FR},2}}\arctan\left(\sqrt{BT_{\mathrm{FR},2}}\right)\left(T_{\mathrm{FR},2} - T + \rho_1^{\mathrm{e}} T\right)}{2\left(T_{\mathrm{FR},2} - T\right)}.$$

$$p_{\mathrm{FFR},2}^{\mathrm{i}}(T) = \frac{\lambda_2\left(1 + \rho_2\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{T_{\mathrm{FR},2}}\right)\right) + \lambda_1\sqrt{P_1/P_2}\left(\sqrt{1/B} + \rho_1\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{BT_{\mathrm{FR},2}}\right)\right)}{\lambda_2\left(1 + 2\xi\left(\rho_2^{\mathrm{i}}, \rho_2, T, T_{\mathrm{FR},2}, 4\right)\right) + \lambda_1\sqrt{P_1/P_2B}\left(1 + 2\xi\left(\rho_1^{\mathrm{e}}, \rho_1, BT, BT_{\mathrm{FR},2}, 4\right)\right)} \tag{27}$$

$$p_{\mathrm{FFR},2}^{\mathrm{e}}(T) = \frac{\frac{2\pi\lambda_2}{\mathcal{A}_2}\int_0^\infty x\exp\left(-\frac{T}{\mathrm{SNR}_2(x,1)}\right)\exp\left\{-\pi\lambda_2 Q(T,\alpha,1,\rho_2^{\mathrm{e}})x^2 - \pi\lambda_1\left(\frac{P_1}{P_2}\right)^{\frac{2}{\alpha}} Q(\beta T,\alpha,1/B,\rho_1^{\mathrm{i}})x^2\right\}dx}{1 - p_{c,2}(T_{\mathrm{FR},2})} - \frac{\frac{2\pi\lambda_2}{\mathcal{A}_2}\int_0^\infty x\exp\left(-\frac{T+T_{\mathrm{FR},2}}{\mathrm{SNR}_2(x,1)}\right)\exp\left\{-\pi\lambda_2 x^2\left[1 + 2\xi\left(\rho_2^{\mathrm{e}}, \rho_2, T, T_{\mathrm{FR},2}, \alpha\right)\right] - \pi\lambda_1\left(\frac{P_1}{P_2B}\right)^{\frac{2}{\alpha}} x^2\left[1 + 2\xi\left(\rho_1^{\mathrm{i}}, \rho_1, \beta BT, BT_{\mathrm{FR},2}, \alpha\right)\right]\right\}dx}{1 - p_{c,2}(T_{\mathrm{FR},2})} \tag{28}$$

$$p_{\mathrm{FFR},2}^{\mathrm{e}}(T) = \frac{\left(\lambda_1\sqrt{P_1/P_2B} + \lambda_2\right)\left[\lambda_2\left(1 + \rho_2\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{T_{\mathrm{FR},2}}\right)\right) + \lambda_1\sqrt{P_1/P_2}\left(\sqrt{1/B} + \rho_1\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{BT_{\mathrm{FR},2}}\right)\right)\right]}{\left(1 - \lambda_1\sqrt{P_1/P_2B} - \lambda_2\right)\left[\lambda_2\left(1 + \rho_2^{\mathrm{e}}\sqrt{T}\arctan\left(\sqrt{T}\right)\right) + \lambda_1\sqrt{P_1/P_2}\left(\sqrt{1/B} + \rho_1^{\mathrm{i}}\sqrt{\beta T}\arctan\left(\sqrt{\beta BT}\right)\right)\right]} - \frac{\left(\lambda_1\sqrt{P_1/P_2B} + \lambda_2\right)\left[\lambda_2\left(1 + \rho_2\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{T_{\mathrm{FR},2}}\right)\right) + \lambda_1\sqrt{P_1/P_2}\left(\sqrt{1/B} + \rho_1\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{BT_{\mathrm{FR},2}}\right)\right)\right]}{\left(1 - \lambda_1\sqrt{P_1/P_2B} - \lambda_2\right)\left[\lambda_2\left(1 + 2\xi\left(\rho_2^{\mathrm{e}}, \rho_2, T, T_{\mathrm{FR},2}, 4\right)\right) + \lambda_1\sqrt{P_1/P_2B}\left(1 + 2\xi\left(\rho_1^{\mathrm{i}}, \rho_1, \beta BT, BT_{\mathrm{FR},2}, 4\right)\right)\right]} \tag{29}$$

When $T = T_{\mathrm{FR},2}$, the limit $T \to T_{\mathrm{FR},2}$ will simplify as $\xi\left(\rho_2^{\mathrm{i}}, \rho_2, T, T_{\mathrm{FR},2}, 4\right) =$

$$\frac{\left(2\rho_2 + 2\rho_2^{\mathrm{i}} - \rho_2\rho_2^{\mathrm{i}}\right)\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{T_{\mathrm{FR},2}}\right)}{4} + \frac{\rho_2\rho_2^{\mathrm{i}} T_{\mathrm{FR},2}}{4\left(T_{\mathrm{FR},2}+1\right)},$$

and $\xi\left(\rho_1^{\mathrm{e}}, \rho_1, BT, BT_{\mathrm{FR},2}, 4\right) =$

$$\frac{\left(2\rho_1 + 2\rho_1^{\mathrm{e}} - \rho_1\rho_1^{\mathrm{e}}\right)\sqrt{BT_{\mathrm{FR},2}}\arctan\left(\sqrt{BT_{\mathrm{FR},2}}\right)}{4} + \frac{\rho_1\rho_1^{\mathrm{e}} BT_{\mathrm{FR},2}}{4\left(BT_{\mathrm{FR},2}+1\right)}.$$

Theorem 5 *(Pico tier, edge user): the coverage probability of the pico edge user is given by* (28).

Proof A user $u \in \mathcal{U}_2$ with SINR $< T_{\mathrm{FR},2}$ is allocated a sub-channel from available sub-channel subset $\mathbf{C}_1$. Conditioning on its previous SINR and applying the Bayes' rule, we have $p_{\mathrm{FFR},2}^{\mathrm{e}}(T)$

$$= \mathbb{P}\left(\frac{P_2\hat{H}_x x^{-\alpha}}{\beta\hat{I}_{x,1}^{\mathrm{i}} + \hat{I}_{x,2}^{\mathrm{e}} + \sigma^2} > T \,\middle|\, \frac{P_2 H_x x^{-\alpha}}{I_{x,1} + I_{x,2} + \sigma^2} < T_{\mathrm{FR},2}\right)$$

$$= \frac{\mathbb{P}\left(\frac{P_2\hat{H}_x x^{-\alpha}}{\beta\hat{I}_{x,1}^{\mathrm{i}} + \hat{I}_{x,2}^{\mathrm{e}} + \sigma^2} > T,\ \frac{P_2 H_x x^{-\alpha}}{I_{x,1} + I_{x,2} + \sigma^2} < T_{\mathrm{FR},2}\right)}{\mathbb{P}\left(\frac{P_2 H_x x^{-\alpha}}{I_{x,1} + I_{x,2} + \sigma^2} < T_{\mathrm{FR},2}\right)}$$

Following the methods of Theorems 1 and 3 give the desired result. □

Corollary 4 *For $\alpha = 4$ and $\sigma^2 = 0$, the coverage probability of the pico edge user is given by* (29)*, where* $\xi\left(\rho_2^{\mathrm{e}}, \rho_2, T, T_{\mathrm{FR},2}, 4\right) =$

$$\frac{\rho_2^{\mathrm{e}}\sqrt{T}\arctan\left(\sqrt{T}\right)\left(T_{\mathrm{FR},2} - T - \rho_2 T_{\mathrm{FR},2}\right)}{2\left(T_{\mathrm{FR},2} - T\right)} + \frac{\rho_2\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{T_{\mathrm{FR},2}}\right)\left(T_{\mathrm{FR},2} - T + \rho_2^{\mathrm{e}} T\right)}{2\left(T_{\mathrm{FR},2} - T\right)},$$

and $\xi\left(\rho_1^{\mathrm{i}}, \rho_1, \beta BT, BT_{\mathrm{FR},2}, 4\right) =$

$$\frac{\rho_1^{\mathrm{i}}\sqrt{\beta BT}\arctan\left(\sqrt{\beta BT}\right)\left(T_{\mathrm{FR},2} - \beta T - \rho_1 T_{\mathrm{FR},2}\right)}{2\left(T_{\mathrm{FR},2} - \beta T\right)} + \frac{\rho_1\sqrt{BT_{\mathrm{FR},2}}\arctan\left(\sqrt{BT_{\mathrm{FR},2}}\right)\left(T_{\mathrm{FR},2} - \beta T + \rho_1^{\mathrm{i}} T\right)}{2\left(T_{\mathrm{FR},2} - \beta T\right)}.$$

When $T = T_{\mathrm{FR},2}$, the limit $T \to T_{\mathrm{FR},2}$ will simplify as $\xi\left(\rho_2^{\mathrm{e}}, \rho_2, T, T_{\mathrm{FR},2}, 4\right) =$

$$\frac{\left(2\rho_2 + 2\rho_2^{\mathrm{e}} - \rho_2\rho_2^{\mathrm{e}}\right)\sqrt{T_{\mathrm{FR},2}}\arctan\left(\sqrt{T_{\mathrm{FR},2}}\right)}{4} + \frac{\rho_2\rho_2^{\mathrm{e}} T_{\mathrm{FR},2}}{4\left(T_{\mathrm{FR},2}+1\right)},$$

and $\xi\left(\rho_1^{\mathrm{i}}, \rho_1, \beta BT, BT_{\mathrm{FR},2}, 4\right) =$

$$\frac{\rho_1^{\mathrm{i}}(1-\beta-\rho_1)\sqrt{\beta BT_{\mathrm{FR},2}}\arctan\left(\sqrt{\beta BT_{\mathrm{FR},2}}\right)}{2(1-\beta)} + \frac{\rho_1\left(1-\beta+\rho_1^{\mathrm{i}}\right)\sqrt{BT_{\mathrm{FR},2}}\arctan\left(\sqrt{BT_{\mathrm{FR},2}}\right)}{2(1-\beta)}.$$

3.2 Average user rate

Similar to definition of the conditional coverage probabilities, the average rate for a typical user is given as

$$\mathcal{R} = \sum_{k=1}^{2}\mathcal{A}_k\left(p_{c,k}\left(T_{\mathrm{FR},k}\right)\mathcal{R}_{\mathrm{FFR},k}^{\mathrm{i}} + \left(1 - p_{c,k}\left(T_{\mathrm{FR},k}\right)\right)\mathcal{R}_{\mathrm{FFR},k}^{\mathrm{e}}\right). \tag{30}$$

where $\mathcal{R}_{\mathrm{FFR},k}^{\mathrm{i}}$ and $\mathcal{R}_{\mathrm{FFR},k}^{\mathrm{e}}$ are the average rate of a typical user when it is an interior user and edge user associated with the kth tier BS, respectively.

The average user rate is computed in units of nats/sec/Hz (1 bit $= \ln(2) = 0.693$ nats), where it can represent the spectral efficiency of a user. Conditioning on a typical user at a distance x from its serving BS in the kth tier, $\mathcal{R}_{\mathrm{FFR},k}^{\mathrm{i}}$ and $\mathcal{R}_{\mathrm{FFR},k}^{\mathrm{e}}$ are thus defined as

$$\mathcal{R}_{\mathrm{FFR},k}^{\mathrm{i}} \triangleq \mathbb{E}_x\left[\mathbb{E}_{\mathrm{SINR}}\left[t_k^{\mathrm{i}}\ln(1+\mathrm{SINR})\,\middle|\, u \in \mathcal{U}_k^{\mathrm{i}}\right]\right], \tag{31}$$

$$\mathcal{R}_{\mathrm{FFR},k}^{\mathrm{e}} \triangleq \mathbb{E}_x\left[\mathbb{E}_{\mathrm{SINR}}\left[t_k^{\mathrm{e}}\ln(1+\mathrm{SINR})\,\middle|\, u \in \mathcal{U}_k^{\mathrm{e}}\right]\right]. \tag{32}$$

where the expectation is taken with respect to the distance x, t_k^{i}, and t_k^{e} are the faction of time the users $u \in \mathcal{U}_k^{\mathrm{i}}$ and $u \in \mathcal{U}_k^{\mathrm{e}}$ are served on a sub-channel, respectively. t_k^{i} and t_k^{e} can be considered as the reciprocal of average number of users sharing a given sub-channel, i.e., the load at a sub-channel.

Lemma 4 *The expectations of the faction of time on a given sub-channel $\mathbb{E}\left[t_k^{\mathrm{i}}\right]$ and $\mathbb{E}\left[t_k^{\mathrm{e}}\right]$ are given by* (33–36)*.*

$$\mathbb{E}\left[t_1^{\mathrm{i}}\right] = \frac{\lambda_1|\mathbf{C}_1|}{\lambda_u\mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)}\left[1 - \left(1 + \frac{\lambda_u\mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)}{3.5\lambda_1}\right)^{-3.5}\right] + \sum_{n=0}^{|\mathbf{C}_1|-1}\left(1 - \frac{|\mathbf{C}_1|}{n+1}\right)\mathbb{P}\left(\tilde{N}_1^{\mathrm{i}} = n\right) \tag{33}$$

$$\mathbb{E}\left[t_1^{\mathrm{e}}\right] = \frac{\lambda_1|\mathbf{C}_2|}{\lambda_u\mathcal{A}_1\left(1 - p_{c,1}\left(T_{\mathrm{FR},1}\right)\right)}\left[1 - \left(1 + \frac{\lambda_u\mathcal{A}_1\left(1 - p_{c,1}\left(T_{\mathrm{FR},1}\right)\right)}{3.5\lambda_1}\right)^{-3.5}\right] + \sum_{n=0}^{|\mathbf{C}_2|-1}\left(1 - \frac{|\mathbf{C}_2|}{n+1}\right)\mathbb{P}\left(\tilde{N}_1^{\mathrm{e}} = n\right) \tag{34}$$

$$\mathbb{E}\left[t_2^{\mathrm{i}}\right] = \frac{\lambda_2|\mathbf{C}_2|}{\lambda_u\mathcal{A}_2 p_{c,2}\left(T_{\mathrm{FR},2}\right)}\left[1 - \left(1 + \frac{\lambda_u\mathcal{A}_2 p_{c,2}\left(T_{\mathrm{FR},2}\right)}{3.5\lambda_2}\right)^{-3.5}\right] + \sum_{n=0}^{|\mathbf{C}_2|-1}\left(1 - \frac{|\mathbf{C}_2|}{n+1}\right)\mathbb{P}\left(\tilde{N}_2^{\mathrm{i}} = n\right) \tag{35}$$

$$\mathbb{E}\left[t_2^{\mathrm{e}}\right] = \frac{\lambda_2 \left|\mathbf{C}_1\right|}{\lambda_u \mathcal{A}_2 \left(1 - p_{c,2}\left(T_{\mathrm{FR},2}\right)\right)} \left[1 - \left(1 + \frac{\lambda_u \mathcal{A}_2 \left(1 - p_{c,2}\left(T_{\mathrm{FR},2}\right)\right)}{3.5\lambda_2}\right)^{-3.5}\right] + \sum_{n=0}^{|\mathbf{C}_1|-1} \left(1 - \frac{|\mathbf{C}_1|}{n+1}\right) \mathbb{P}\left(\tilde{N}_2^{\mathrm{e}} = n\right) \tag{36}$$

Proof The proof is given in Appendix D. □

Assuming the independence between the load and SINR [14, 15, 23], the following theorem gives the average rate for a typical macro interior user. Note that the average rate for other type of users can be obtained following the same procedure.

Theorem 6 *The average rate of a typical macro interior user is*

$$\mathcal{R}^{\mathrm{i}}_{\mathrm{FFR},1} = \mathbb{E}\left[t_1^{\mathrm{i}}\right] \times \int_0^\infty p^{\mathrm{i}}_{\mathrm{FFR},1}\left(e^t - 1\right) dt. \tag{37}$$

Proof Following the assumption of independence between the load and SINR, we have

$$\mathcal{R}^{\mathrm{i}}_{\mathrm{FFR},1} = \mathbb{E}\left[t_1^{\mathrm{i}}\right] \times \mathbb{E}_x\left[\mathbb{E}_{\mathrm{SINR}}\left[\ln\left(1+\mathrm{SINR}\right) \middle| u \in \mathcal{U}_1^{\mathrm{i}}\right]\right]. \tag{38}$$

Since $\mathbb{E}[\tau] = \int_0^\infty \mathbb{P}(\tau > t)\, dt$ for $\tau > 0$, we obtain □

$$\begin{aligned}
&\mathbb{E}_x\left[\mathbb{E}_{\mathrm{SINR}}\left[\ln\left(1+\mathrm{SINR}\right) \middle| u \in \mathcal{U}_1^{\mathrm{i}}\right]\right] \\
&= \int_0^\infty \mathbb{P}\left(\ln\left(1+\mathrm{SINR}\right) > t \middle| u \in \mathcal{U}_1^{\mathrm{i}}\right) dt \\
&= \int_0^\infty \mathbb{P}\left(\ln\left(1 + \frac{\beta P_1 \hat{H}_x x^{-\alpha}}{\beta \hat{I}^{\mathrm{i}}_{x,1} + \hat{I}^{\mathrm{e}}_{x,2} + \sigma^2}\right) > t \middle| \frac{P_1 H_x x^{-\alpha}}{I_{x,1} + I_{x,2} + \sigma^2} > T_{\mathrm{FR},1}\right) dt. \\
&= \int_0^\infty \mathbb{P}\left(\frac{\beta P_1 \hat{H}_x x^{-\alpha}}{\beta \hat{I}^{\mathrm{i}}_{x,1} + \hat{I}^{\mathrm{e}}_{x,2} + \sigma^2} > e^t - 1 \middle| \frac{P_1 H_x x^{-\alpha}}{I_{x,1} + I_{x,2} + \sigma^2} > T_{\mathrm{FR},1}\right) dt
\end{aligned}$$

Conditioning on x the distance to its serving macro BS and following the method of Theorem 2, we have

$$\mathbb{E}_x\left[\mathbb{E}_{\mathrm{SINR}}\left[\ln\left(1+\mathrm{SINR}\right) \middle| u \in \mathcal{U}_1^{\mathrm{i}}\right]\right] = \int_0^\infty p^{\mathrm{i}}_{\mathrm{FFR},1}\left(e^t - 1\right) dt.$$

Plugging back into (38), the average rate (37) can be obtained.

4 Simulation and numerical results

In this section, we present numerical results on the coverage probability and average user rate for the proposed SFR scheme. Furthermore, we compare the proposed SFR scheme with FRP scheme. The simulation parameters are in accordance with 3GPP technical reports [28]. Unless otherwise stated, the transmit powers of a macro BS and a pico BS are $P_1^T = 46$ dBm and $P_2^T = 30$ dBm. The densities of the two tiers are $\lambda_1 = 1$ BS/km^2 and $\lambda_2 = 5$ BS/km^2 with $\alpha = 4$. The user density is $\lambda_u = 100$ users/km^2. For a typical LTE system with 10 MHz bandwidth, 50 subchannels (i.e., $|\mathbf{C}| = 50$) are available to each BS, each sub-channel bandwidth then is 200 kHz.

4.1 Validation of analysis

In Fig. 2, the average user rate sweeping over a range of user density λ_u is validated via Monte Carlo methods on a square window of 10×10 km^2. There are three values for the combination of association bias, power control factor and resource partitioning factor (B, β, η) are shown. It can be obtained that the analytical results match quite well with the simulation results. Meanwhile, the simulation results give upper bound with less than a 0.26 dB gap from the analytical results. It also can be observed that the gap decreases with the increase in user density (namely, network load). In addition to the approximation for Voronoi cell areas distribution, the small gap is also

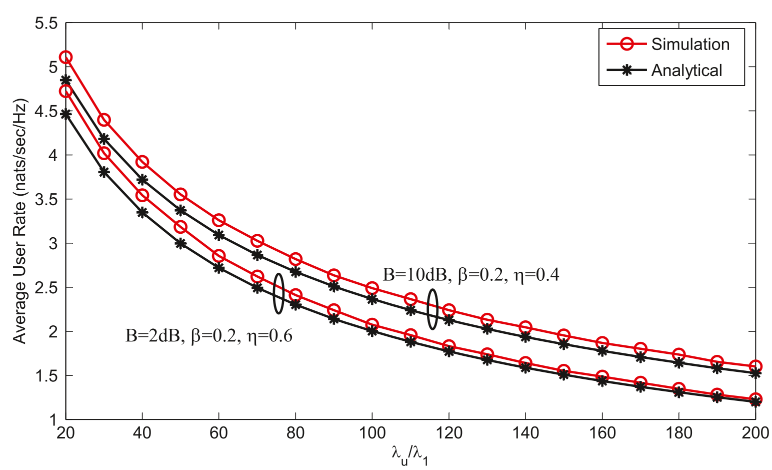

Fig. 2 Validation of the analytical results for average user rate via Monte Carlo methods with $T_{\mathrm{FR},1} = T_{\mathrm{FR},2} = -10$ dB

due to the assumption of the independence between the load and SINR.

4.2 Comparison with FRP scheme

Figure 3 compares the average user rate of the proposed SFR scheme and FRP scheme proposed by [15] in different load conditions. In this paper, the cell load model is dependent on user density and parameters of SFR scheme, apart from association bias. The average user rate of both schemes decreases with the increase in user density. This is because the number of users shared a common resource increases and so does the interference. From this figure, it can be seen that the proposed SFR scheme outperforms the FRP scheme in terms of average user rate in any load condition. The curves indicate that the proposed SFR scheme can enhance the average user rate by about 7.49% when $\lambda_u = 100$ users/km^2. Moreover, the average user rate can be further improved by properly setting up the parameters of SFR scheme.

4.3 Average user rate: trends and discussion

The conditional coverage probabilities without SFR are significant for the classification of users in each tier. Figure 4 shows the variations in the previous cell load ρ_k and $p_{c,k}\left(T_{\mathrm{FR},k}\right)$ with increasing association bias. It can be seen that the previous cell load of macro tier ρ_1 decreases with increasing association bias. So the coverage probability without SFR of macro users $p_{c,1}\left(T_{\mathrm{FR},1}\right)$ increases because the interference from macro tier decreases. On the other hand, despite that the previous cell load of pico tier ρ_2 is directly proportional to the association bias, the coverage probability without SFR of pico users $p_{c,2}\left(T_{\mathrm{FR},2}\right)$ firstly decreases and then increases with increasing association bias. This is because the offloaded users usually do not get the strongest received power and it results in degraded SINR, so the coverage performance of pico tier initially deteriorates. When beyond a certain association bias (10 dB in Fig. 4b), the interference from macro tier for pico users becomes fairly low and also continues to decrease. Then, the aggregate interference decreases and it will results in improved coverage performance.

Figure 5a shows that the conditional probabilities that a user is macro interior user and edge user (macro tier association probabilities) decrease with increasing association bias. Meanwhile, the conditional probability that a user is pico interior user increases with increasing association bias. However, the conditional probability that a user is pico edge user firstly increases and then decreases with increasing association bias. That is, the majority of offloaded users initially become pico edge users owing to their worse SINRs. When beyond a certain association bias, the interference from macro tier becomes fairly low, and thereby decreases its pico edge association probability. Figure 5b shows the variations in the four types of cell loads with SFR with increasing association bias. It can be observed that the variations in the four types of cell loads with SFR are the same as that in Fig. 5a. As η increases, more sub-channels become available for macro interior users and pico edge users, consequently their cell loads ρ_1^{i} and ρ_2^{e} decreases which are different from the cell loads ρ_1^{e} and ρ_2^{i}.

For the sake that the proposed SFR scheme can be better understood, we analyze the conditional coverage probabilities and average rate of four types of users (i.e., the interior and edge users of macro and pico BSs). In Fig. 6, it can be seen that the coverage probability and average rate of macro users (both interior and edge users) increase with increasing association bias. This is because more macro users are offloaded to pico tier and more sub-channels become available for macro users, namely, the

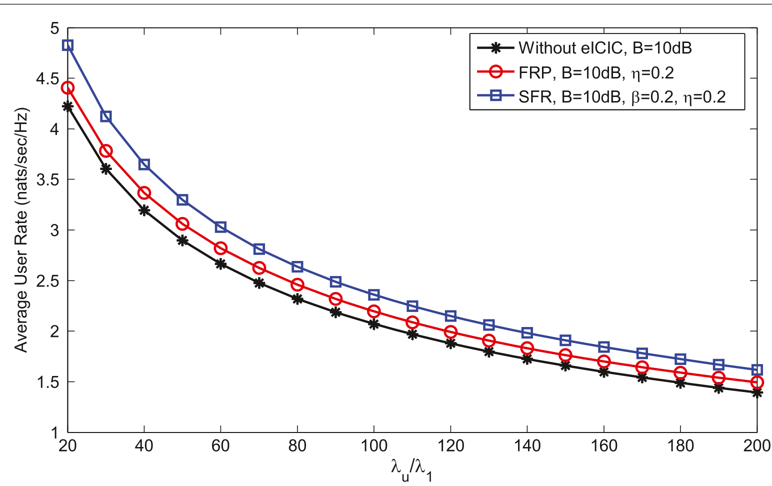

Fig. 3 Comparison of the average user rate of CRE without eICIC, and with FRP and SFR for $T_{\mathrm{FR},1} = T_{\mathrm{FR},2} = -10$ dB

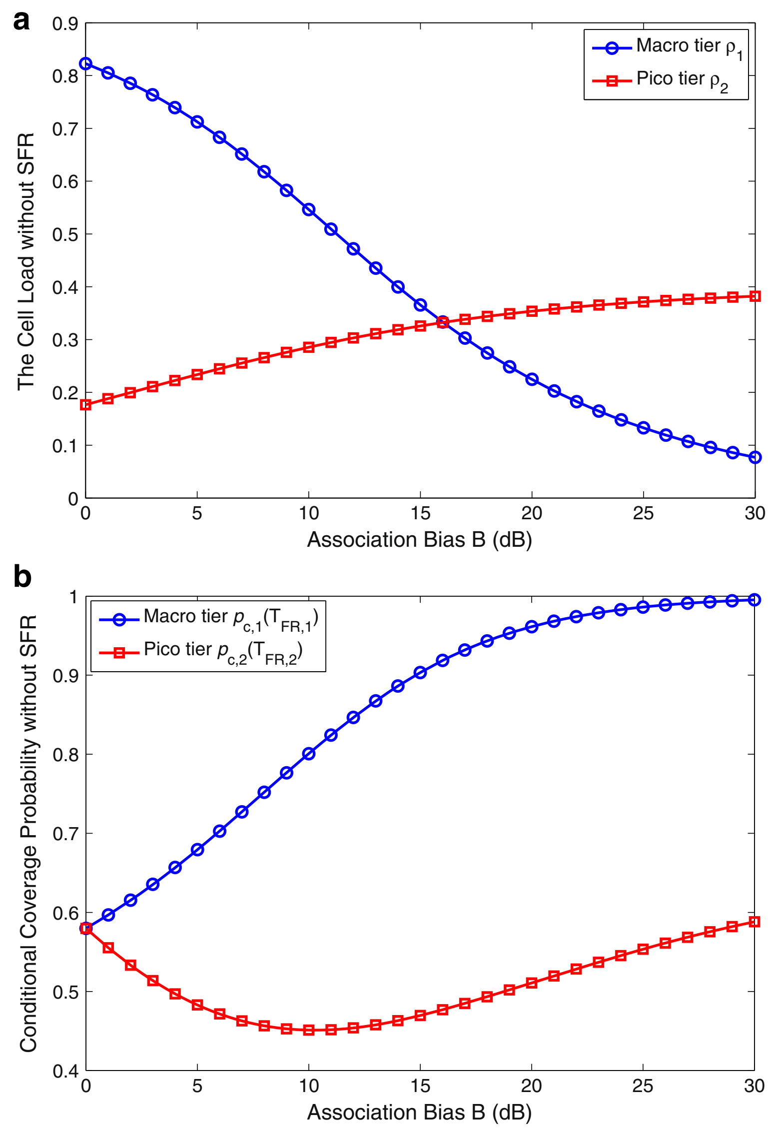

Fig. 4 Effect of the cell loads ρ_1 and ρ_2 on conditional coverage probabilities without SFR $p_{c,1}$ $(T_{FR,1})$ and $p_{c,2}$ $(T_{FR,2})$ for $T_{FR,1} = T_{FR,2} = 3$ dB

decrease in macro loads ρ_1^{i} and ρ_1^{e}. Similar to [15], the variation in the performance with bias of users is mainly due to the change in the cell loads (i.e., ρ_1^{i}, ρ_1^{e}, ρ_2^{i}, and ρ_2^{e}), which are directly related to the interference from each tier. Moreover, there are obviously differences from previous works in terms of the coverage and rate trends of pico edge users, which their performance firstly decreases and then increases with increasing association bias. It is due to that the majority of offloaded users initially become pico edge users owing to their worse SINRs. Hence, the performance of pico edge users initially deteriorates, despite that the interference from macro tier decreases. When beyond a certain association bias, the interference from macro tier becomes fairly low, and thereby the performance of pico edge users increases. But, the performance of pico interior users has no significant change with respect to the increasing association bias. To mitigate the severe interference from macro tier suffered by pico edge users, especially the offloaded macro users, power control (transmit power reduction) is used for macro BSs. It can be obtained in Fig. 6 that the coverage probability and average rate of pico edge users are improved at the expense of the performance degradation of macro interior users. As η increases, more sub-channels become available for pico edge users, consequently their performance can be further improved. However, the performance of macro edge users and pico interior users remains relatively unchanged with regard to the power control.

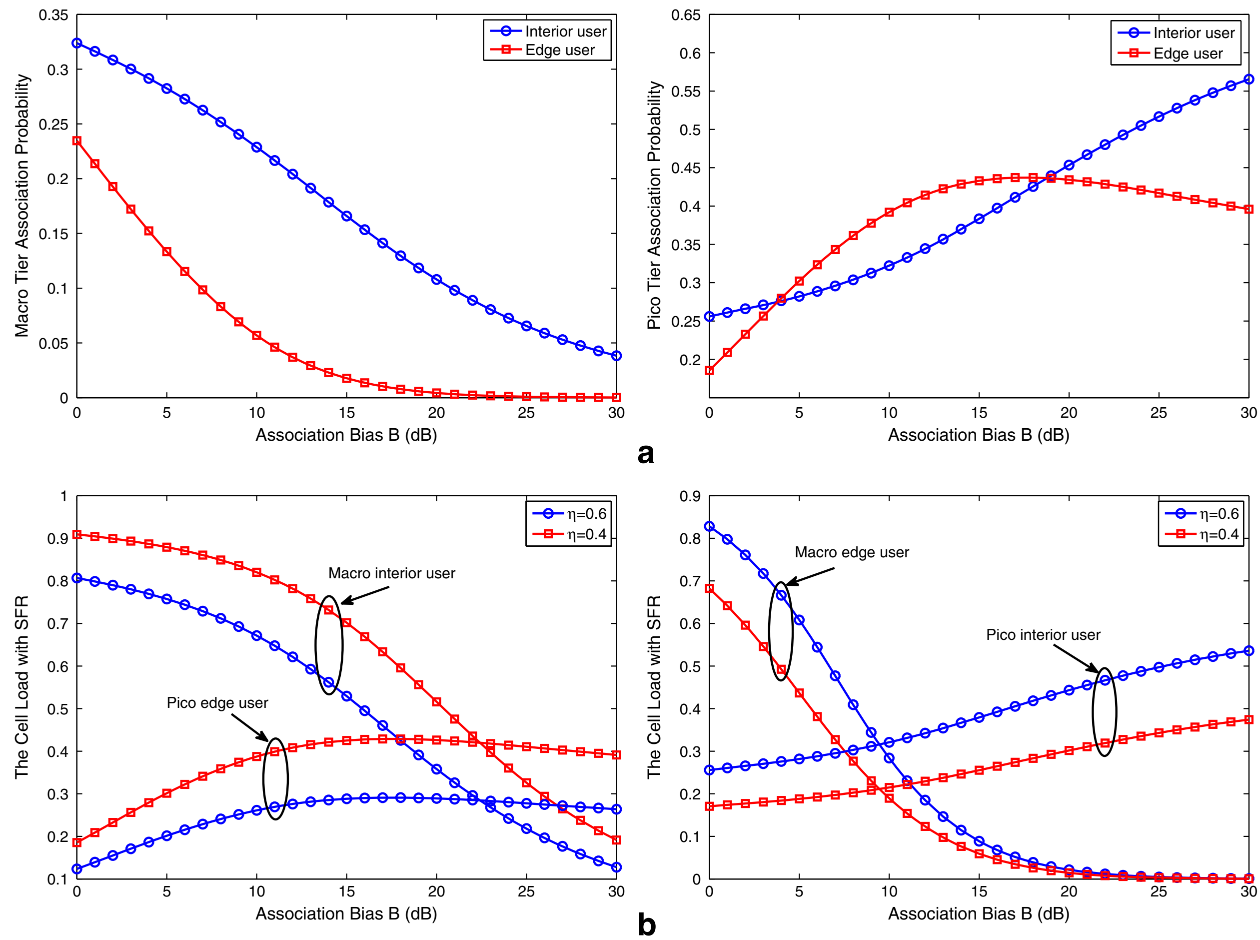

Fig. 5 a The conditional probabilities $\mathcal{A}_k p_{c,k}\left(T_{\mathrm{FR},k}\right)$ and $\mathcal{A}_k\left(1-p_{c,k}\left(T_{\mathrm{FR},k}\right)\right)$ that a user in the kth tier is an interior user and edge user, respectively. **b** The cell loads with SFR for $T_{\mathrm{FR},1}=T_{\mathrm{FR},2}=3$ dB

4.3.1 Impact of SFR thresholds

For the proposed SFR scheme, SFR thresholds $T_{\mathrm{FR},1}$ and $T_{\mathrm{FR},2}$ are important design parameters and play a key role in controlling the gains of pico edge users. The effect of SFR threshold pair $\left(T_{\mathrm{FR},1}, T_{\mathrm{FR},2}\right)$ on average user rate is shown in Fig. 7. It can be observed that the importance of properly dividing interior and edge regions on the average user rate. The optimal pair is $(-20\,\mathrm{dB}, 4\,\mathrm{dB})$ so that the average user rate attains a maximal value of 2.4304 nats/sec/Hz, which gives a significant increase (by around 47.7%) compared to the worst case that just gets 1.6454 nats/sec/Hz with $(20\,\mathrm{dB}, -20\,\mathrm{dB})$. With the macro's SFR threshold $T_{\mathrm{FR},1}$ decreases gradually from 20 dB to − 20 dB and the matched pico's SFR threshold $T_{\mathrm{FR},2}$ increases from − 20 dB to 20 dB, the average user rate initially increases, but decreases beyond a certain SFR threshold pair and hence, the optimal SFR threshold pair exists. For the given value of association bias ($B = 10$ dB), the total number of pico users is much greater than the total number of macro users. Therefore, the variation in the overall average user rate is mainly contributed by pico users. As $T_{\mathrm{FR},2}$ increases, the number of pico edge users increases while the number of pico interior users decreases. On the contrary, the number of macro interior users increase and the number of macro edge users decrease with the decrease in $T_{\mathrm{FR},1}$. With the variation in SFR threshold pair, the average rate of a typical pico interior user decreases because the increase in shared resources and the decrease in interference from macro tier. Meanwhile, the average rate of a typical pico edge user varies inversely. Thus, the contribution of pico interior users towards the average rate initially dominates. But after a certain SFR threshold pair, the contribution of pico edge users towards the average rate eventually dominates. Furthermore, the main interference resulting in the degradation of network performance is the strong interference from macro tier. Hence, the macro interior regions tend to be larger because of the introduced power control can mitigate such strong interference.

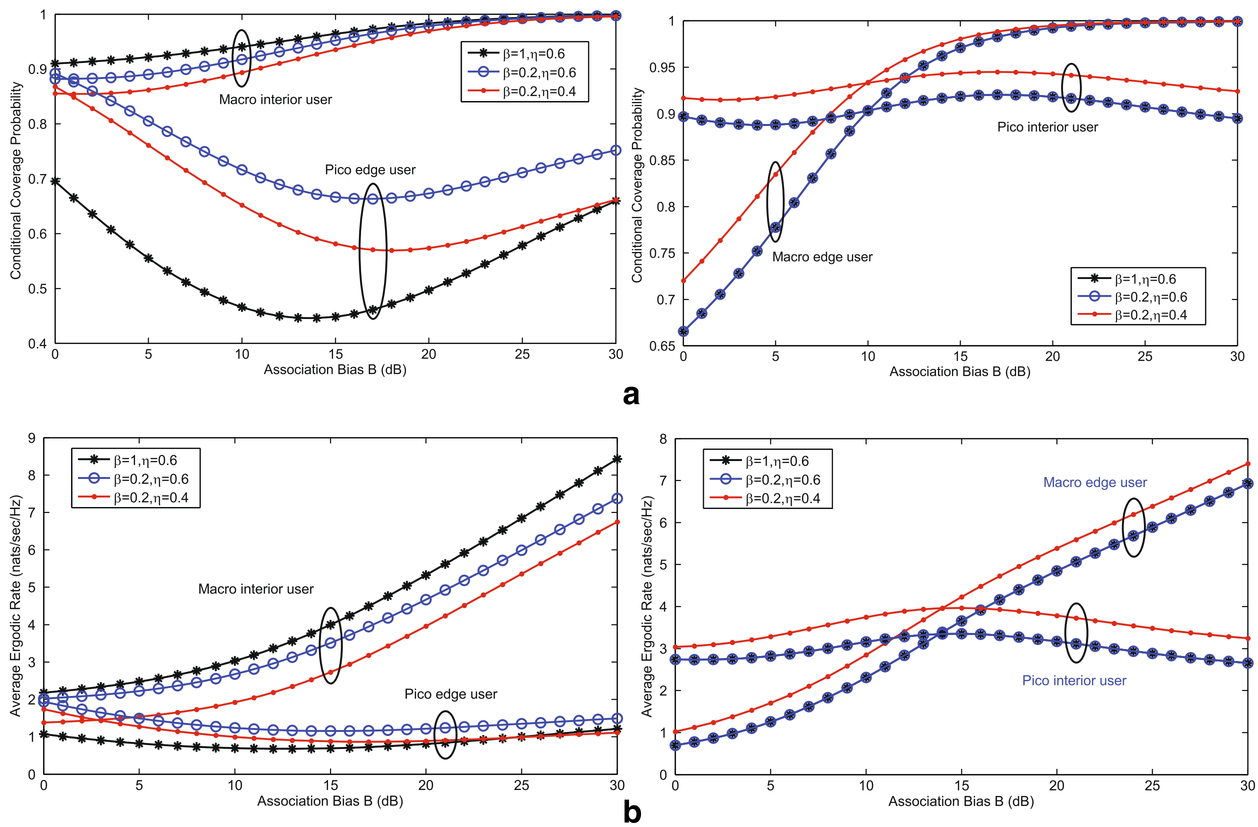

Fig. 6 Effect of power control factor and resource partitioning factor (β, η) on conditional coverage probabilities and average rate of four types of user with **a** $T_{\mathrm{FR},1} = T_{\mathrm{FR},2} = 3$ dB and $T = 0.5$ (-3 dB) **b** $T_{\mathrm{FR},1} = T_{\mathrm{FR},2} = 3$ dB, as association bias is varied

4.3.2 Impact of power control

Figure 8 shows the effect of association bias and power control factor on average user rate. It can be seen that the average user rate can be significantly improved by introducing power control for macro interior users. With increasing power control factor, the average user rate initially increases, but decreases beyond a certain power control factor and hence, the optimal power control factor exists. Note that the average rate of macro edge users and pico interior users are invariant to power control factor. Meanwhile, with decreasing power control factor, the increase in average rate of pico edge users dominates in comparison with the decrease in average rate of macro interior users. Thus, the overall average rate is improved.

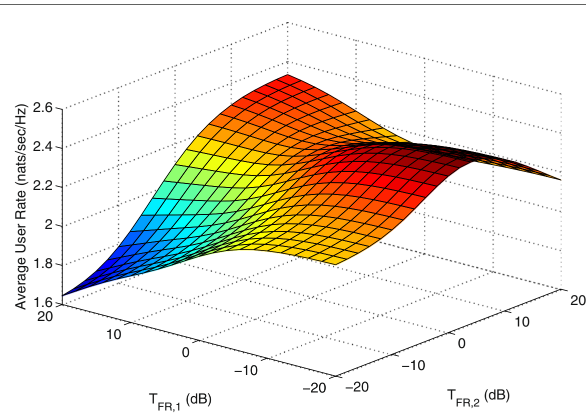

Fig. 7 Effect of SFR thresholds $T_{\mathrm{FR},1}$ and $T_{\mathrm{FR},2}$ on average user rate with $B = 10$ dB, $\beta = 0.2$, and $\eta = 0.6$

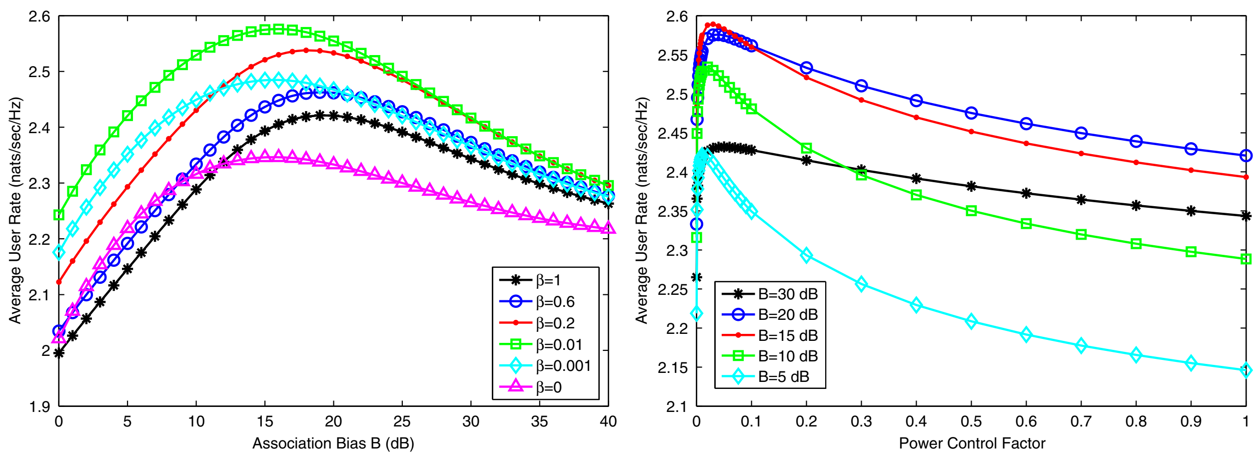

Fig. 8 Effect of association bias and power control factor on average user rate with $\eta = 0.6$, $T_{\mathrm{FR},1} = -20$ dB, and $T_{\mathrm{FR},2} = 4$ dB

But after a certain power control factor, the average rate of macro interior users is so small that the overall average rate decreases. It can be obviously observed that the optimal power control factor tends to be smaller because of the large transmit power of macro BS.

For a given power control factor, the average user rate initially increases as the association bias increases, but decreases beyond a certain association bias. Hence, the optimal association bias exists. This is because more macro users in poor situation is offloaded to pico tier with the increase in association bias and thus, the interference from macro tier decreases due to the macro cell load decreases. This leads to the average user rate initially increases. But after a certain association bias, the pico BS is overloaded so that the average user rate eventually decreases.

4.3.3 Impact of resource partitioning

Figure 9 shows the effect of association bias and resource partitioning factor on average user rate. For a given resource partitioning factor, the average user rate initially increases with increasing association bias, but decreases beyond a certain association bias. The reason of variation trend was already discussed above. For a given association bias, the average user rate initially increases with increasing resource partitioning factor, but decreases beyond a certain resource partitioning factor. This is because, more sub-channels become available for macro interior users and pico edge users as resource partitioning factor increases, then the corresponding cell load and interference decreases. Consequently, their average user rate increases. The average rate of macro edge users and pico interior users decreases because less sub-channels are allocated to them, resulting in their cell load and interference increases. Thus, the contribution of macro interior users and pico edge users towards the average rate initially dominates. But after a certain resource partitioning factor, the contribution of macro edge users and pico interior users towards the average rate eventually dominates. Furthermore, the resource partitioning factor is tightly

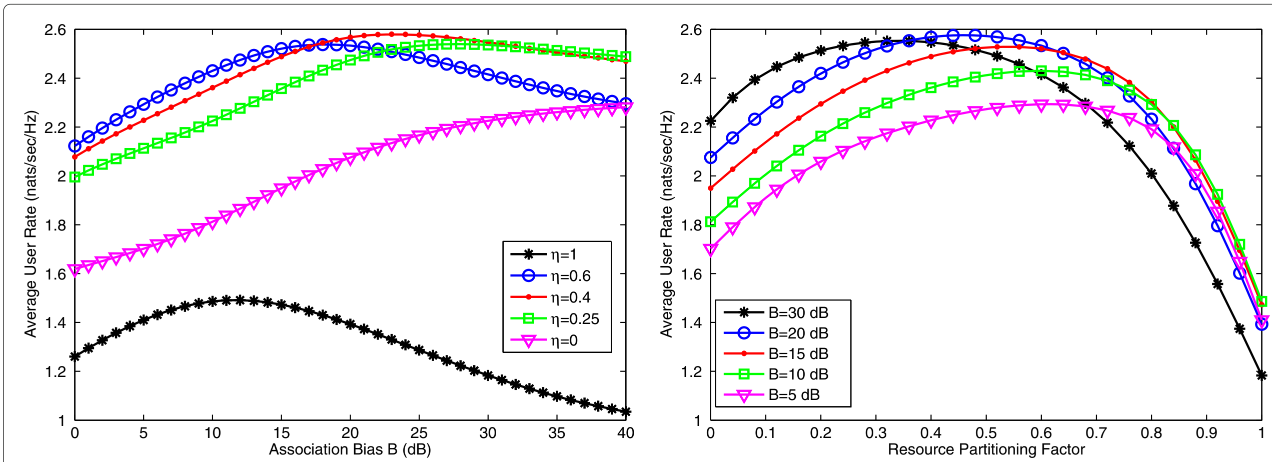

Fig. 9 Effect of association bias and resource partitioning factor on average user rate with $\beta = 0.2$, $T_{\mathrm{FR},1} = -20$ dB, and $T_{\mathrm{FR},2} = 4$ dB

related to association bias and SFR thresholds because of our practical cell load model.

4.3.4 Optimal average user rate

As discussed above, the association bias and parameters of the SFR scheme need to be carefully chosen for optimal average user rate. The average user rate is calculated from the combination of resource partitioning factor and power control factor. However, the results are obtained from varied cell load for each pair. That is, the association bias increases from 0 to 40 dB , and the threshold $T_{\mathrm{FR},1}$ decreases gradually from 20 to -20 dB and the matched threshold $T_{\mathrm{FR},2}$ increases from -20 to 20 dB. Hence, the optimal average user rate is 2.6217 nats/sec/Hz when $B = 18$ dB, $\eta = 0.72$, $\beta = 0.03$, $T_{\mathrm{FR},1} = -20$ dB, and $T_{\mathrm{FR},2} = 12$ dB. Compared to the network without CRE that just gets 1.7704 nats/sec/Hz, the gain obtained from the CRE with SFR scheme can be as high as 48.1%.

5 Conclusions

In this paper, we have presented an analytical framework to evaluate average user rate of HCNs with CRE and SFR scheme in a multichannel environment using the tools of stochastic geometry. Taking the random resource allocation and cell load into consideration, our network model can adequately capture the impact of CRE and SFR scheme on the performance. The numerical results have shown that the average user rate can be improved significantly, and further, the proposed SFR scheme outperforms the FRP scheme in any load condition. In addition, the optimal average user rate can be achieved by properly tuning the association bias and parameters of the SFR scheme. With the optimal combination, the gain can be as high as 48.1%.

Appendix A

Proof of Theorem 1 The conditional coverage probability without SFR of user $u \in \mathcal{U}_k$ is

$$p_{c,k}(T) = \int_{x=0}^{\infty} \mathbb{P}\left(\mathrm{SINR} > T \,|u \in \mathcal{U}_k, X_k = x\right) f_{X_k}(x)\, dx. \tag{39}$$

□

Using the SINR expression in (6), we have

$$\begin{aligned}
\mathbb{P}\left(\mathrm{SINR} > T \,|u \in \mathcal{U}_1, X_1 = x\right) &= \mathbb{P}\left(\frac{P_1 H_x x^{-\alpha}}{\sum_{j=1}^{2} I_{x,j} + \sigma^2} > T\right) \\
&= \mathbb{P}\left(H_x > x^{\alpha} P_1^{-1} T \left\{\sum\nolimits_{j=1}^{2} I_{x,j} + \sigma^2\right\}\right) \\
&\overset{(a)}{=} \mathbb{E}_{I_{x,j}}\left[\exp\left(-x^{\alpha} P_1^{-1} T \left\{\sum\nolimits_{j=1}^{2} I_{x,j} + \sigma^2\right\}\right)\right] \\
&\overset{(b)}{=} \exp\left(-\frac{T}{\mathrm{SNR}_1(x)}\right) \prod_{j=1}^{2} \mathbb{E}_{I_{x,j}}\left[\exp\left(-x^{\alpha} P_1^{-1} T I_{x,j}\right)\right] \\
&= \exp\left(-\frac{T}{\mathrm{SNR}_1(x)}\right) \prod_{j=1}^{2} \mathcal{L}_{I_{x,j}}\left(x^{\alpha} P_1^{-1} T\right)
\end{aligned} \tag{40}$$

where $\mathrm{SNR}_1(x) = \frac{P_1 x^{-\alpha}}{\sigma^2}$, (a) follows that $H_x \sim \exp(1)$, (b) follows from the independence of $I_{x,j}$ and $\mathcal{L}_{I_{x,j}}(s)$ is the Laplace transform of interference. Similarly

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{SINR} > T \,|u \in \mathcal{U}_2, X_2 = x\right) \\
&\quad = \exp\left(-\tfrac{T}{\mathrm{SNR}_2(x)}\right) \prod_{j=1}^{2} \mathcal{L}_{I_{x,j}}\left(x^{\alpha} P_2^{-1} T\right).
\end{aligned} \tag{41}$$

Considering the definition of the Laplace transform and the serving BS being b_k, we have

$$\begin{aligned}
\mathcal{L}_{I_{x,j}}(s) &= \mathbb{E}_{I_{x,j}}\left[\exp\left(-sI_{x,j}\right)\right] \\
&= \mathbb{E}_{\Psi_j,\{H_y\}}\left[\exp\left(-sP_j \sum_{y \in \Psi_j \backslash b_k} H_y \|y\|^{-\alpha}\right)\right] \\
&\overset{(a)}{=} \mathbb{E}_{\Psi_j}\left[\prod_{y \in \Psi_j \backslash b_k} \mathbb{E}_{H_y}\left[\exp\left(-sP_j H_y \|y\|^{-\alpha}\right)\right]\right] \\
&= \mathbb{E}_{\Psi_j}\left[\prod_{y \in \Psi_j \backslash b_k} \mathcal{L}_{H_y}\left(sP_j \|y\|^{-\alpha}\right)\right] \\
&\overset{(b)}{=} \exp\left(-2\pi \rho_j \lambda_j \int_{d_{jk}(x)}^{\infty} \left(1 - \mathcal{L}_{H_y}\left(sP_j v^{-\alpha}\right)\right) v\, dv\right) \\
&\overset{(c)}{=} \exp\left(-2\pi \rho_j \lambda_j \int_{d_{jk}(x)}^{\infty} \frac{v}{1 + \left(sP_j\right)^{-1} v^{\alpha}} dv\right)
\end{aligned}$$

where (a) is obtained from the independence of H_y, (b) follows from the probability generating functional (PGFL) [29] of Ψ_j and replacing $v = \|y\|$, and (c) follows that $H_y \sim \exp(1)$. Moreover, $d_{jk}(x)$ is the lower bound on distance of the jth tier when the typical user $u \in \mathcal{U}_k$, which can be obtained by using (1) as

$$\begin{aligned}
&\text{if } k = 1: d_{11} = x, d_{21} = \left(P_2 B / P_1\right)^{1/\alpha} x \\
&\text{if } k = 2: d_{12} = \left(P_1 / P_2 B\right)^{1/\alpha} x, d_{22} = x.
\end{aligned}$$

Using the change of variables $\left(sP_j\right)^{-2/\alpha} v^2 \to u$, the integral can be simplified as

$$\begin{aligned}
&\int_{d_{jk}(x)}^{\infty} \frac{v}{1 + \left(sP_j\right)^{-1} v^{\alpha}} dv \\
&\quad = \frac{1}{2}\left(sP_j\right)^{2/\alpha} \int_{\left(sP_j\right)^{-2/\alpha}\left(d_{jk}(x)\right)^2}^{\infty} \frac{du}{1 + u^{\alpha/2}}. \\
&\quad = \frac{1}{2}\left(sP_j\right)^{2/\alpha} Z\left(1, \alpha, \frac{\left(d_{jk}(x)\right)^{\alpha}}{sP_j}\right)
\end{aligned}$$

Hence, the Laplace transform of interference is

$$\mathcal{L}_{I_{x,j}}(s) = \exp\left(-\pi \rho_j \lambda_j \left(sP_j\right)^{2/\alpha} Z\left(1, \alpha, \frac{\left(d_{jk}(x)\right)^{\alpha}}{sP_j}\right)\right), \quad (42)$$

where $Z(a,b,c) = a^{2/b} \int_{\left(\frac{c}{a}\right)^{2/b}}^{\infty} \frac{du}{1+u^{b/2}}$.

Using (3) in (39) along with (40–42), the conditional coverage probabilities given in Theorem 1 are obtained.

Appendix B

Proof of Theorem 2 A user $u \in \mathcal{U}_1$ with SINR $> T_{\mathrm{FR},1}$ is allocated a sub-channel from available sub-channel subset $\mathbf{C}_1$ and will experience new fading power $\hat{H}_x$ and out-of-cell interference from both tiers, instead of H_x and $\sum_{j=1}^{2} I_{x,j}$. Conditioning on its previous SINR, the coverage probability of the macro interior user is

$$p^{\mathrm{i}}_{\mathrm{FFR},1}(T) = \mathbb{P}\left(\frac{\beta P_1 \hat{H}_x x^{-\alpha}}{\beta \hat{I}^{\mathrm{i}}_{x,1} + \hat{I}^{\mathrm{e}}_{x,2} + \sigma^2} > T \,\middle|\, \frac{P_1 H_x x^{-\alpha}}{I_{x,1} + I_{x,2} + \sigma^2} > T_{\mathrm{FR},1}\right) \quad (43)$$

□

Following the Bayes' rule, we have

$$p^{\mathrm{i}}_{\mathrm{FFR},1}(T) = \frac{\mathbb{P}\left(\frac{\beta P_1 \hat{H}_x x^{-\alpha}}{\beta \hat{I}^{\mathrm{i}}_{x,1} + \hat{I}^{\mathrm{e}}_{x,2} + \sigma^2} > T, \frac{P_1 H_x x^{-\alpha}}{I_{x,1}+I_{x,2}+\sigma^2} > T_{\mathrm{FR},1}\right)}{\mathbb{P}\left(\frac{P_1 H_x x^{-\alpha}}{I_{x,1}+I_{x,2}+\sigma^2} > T_{\mathrm{FR},1}\right)} \quad (44)$$

$$\overset{(a)}{=} \frac{\mathbb{E}\left[\exp\left(-x^{\alpha} P_1^{-1} T\left(\hat{I}^{\mathrm{i}}_{x,1} + \beta^{-1}\hat{I}^{\mathrm{e}}_{x,2} + \beta^{-1}\sigma^2\right)\right) \times \exp\left(-x^{\alpha} P_1^{-1} T_{\mathrm{FR},1}\left(I_{x,1} + I_{x,2} + \sigma^2\right)\right)\right]}{\mathbb{E}\left[\exp\left(-x^{\alpha} P_1^{-1} T_{\mathrm{FR},1}\left(I_{x,1} + I_{x,2} + \sigma^2\right)\right)\right]}$$

where (a) follows from the independence of $\hat{H}_x$ and H_x, and $\hat{H}_x, H_x \sim \exp(1)$.

Conditioning on x, the distance to its serving macro BS, and focusing on the numerator of (44), we observe that the noise is independence of the interference and the expectation with respect to $\hat{I}^{\mathrm{i}}_{x,1}, I_{x,1}, \hat{I}^{\mathrm{e}}_{x,2}$ and $I_{x,2}$ is the joint Laplace transform $\mathcal{L}\left(\hat{s}_1, s_1, \hat{s}_2, s_2\right)$ of $\hat{I}^{\mathrm{i}}_{x,1}, I_{x,1}, \hat{I}^{\mathrm{e}}_{x,2}$, and $I_{x,2}$ evaluated at $\left(x^{\alpha} P_1^{-1} T, x^{\alpha} P_1^{-1} T_{\mathrm{FR},1}, x^{\alpha}(\beta P_1)^{-1} T, x^{\alpha} P_1^{-1} T_{\mathrm{FR},1}\right)$.

Then, the joint Laplace transform is $\mathcal{L}\left(\hat{s}_1, s_1, \hat{s}_2, s_2\right)$

$$\begin{aligned} &= \mathbb{E}\left[\exp\left(-\hat{s}_1 \hat{I}^{\mathrm{i}}_{x,1} - s_1 I_{x,1} - \hat{s}_2 \hat{I}^{\mathrm{e}}_{x,2} - s_2 I_{x,2}\right)\right] \\ &\overset{(a)}{=} \mathbb{E}\left[\exp\left(-\hat{s}_1 \hat{I}^{\mathrm{i}}_{x,1} - s_1 I_{x,1}\right)\right] \mathbb{E}\left[\exp\left(-\hat{s}_2 \hat{I}^{\mathrm{e}}_{x,2} - s_2 I_{x,2}\right)\right], \end{aligned} \quad (45)$$

where (a) follows from the independence of $\hat{I}^{\mathrm{i}}_{x,1} + I_{x,1}$ and $\hat{I}^{\mathrm{e}}_{x,2} + I_{x,2}$.

Factoring out the first term of (45), we have the derivation of (46), where (a) follows that $\hat{H}_x, H_x \sim \exp(1)$ and (b) is obtained from the probability generating functional (PGFL) [29] of Φ_1, replacing $\nu = \|y\|$ and x is the lower bound on distance between the macro user and its own tier.

$$\begin{aligned} &\mathbb{E}\left[\exp\left(-\hat{s}_1 \hat{I}^{\mathrm{i}}_{x,1} - s_1 I_{x,1}\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\hat{s}_1 P_1 \sum_{y \in \Xi^{\mathrm{i}}_1} \hat{H}_y \|y\|^{-\alpha} - s_1 P_1 \sum_{y \in \Psi_1} H_y \|y\|^{-\alpha}\right)\right] \\ &= \mathbb{E} \prod_{y \in \Phi_1} \exp\left(-\hat{s}_1 P_1 \mathbf{1}\left(y \in \Xi^{\mathrm{i}}_1\right) \hat{H}_y \|y\|^{-\alpha} - s_1 P_1 \mathbf{1}\left(y \in \Psi_1\right) H_y \|y\|^{-\alpha}\right) \\ &= \mathbb{E} \prod_{y \in \Phi_1} \left(1 - \rho^{\mathrm{i}}_1\left(1 - \exp\left(-\hat{s}_1 P_1 \hat{H}_y \|y\|^{-\alpha}\right)\right)\right)\left(1 - \rho_1\left(1 - \exp\left(-s_1 P_1 H_y \|y\|^{-\alpha}\right)\right)\right) \\ &\overset{(a)}{=} \mathbb{E} \prod_{y \in \Phi_1} \left(1 - \rho^{\mathrm{i}}_1\left(1 - \frac{1}{1 + \hat{s}_1 P_1 \|y\|^{-\alpha}}\right)\right)\left(1 - \rho_1\left(1 - \frac{1}{1 + s_1 P_1 \|y\|^{-\alpha}}\right)\right) \\ &\overset{(b)}{=} \exp\left(-2\pi\lambda_1 \int_x^{\infty}\left[1 - \left(1 - \rho^{\mathrm{i}}_1\left(1 - \frac{1}{1 + \hat{s}_1 P_1 \nu^{-\alpha}}\right)\right)\left(1 - \rho_1\left(1 - \frac{1}{1 + s_1 P_1 \nu^{-\alpha}}\right)\right)\right] \mathrm{v}d\mathrm{v}\right) \end{aligned} \quad (46)$$

$$\begin{aligned} &\exp\left(-2\pi\lambda_1 x^2 \int_1^{\infty}\left[1 - \left(1 - \rho^{\mathrm{i}}_1\left(1 - \frac{1}{1 + T\nu^{-\alpha}}\right)\right)\left(1 - \rho_1\left(1 - \frac{1}{1 + T_{\mathrm{FR},1}\nu^{-\alpha}}\right)\right)\right] \mathrm{v}d\mathrm{v}\right) \times \\ &\exp\left(-2\pi\lambda_2 \left(\frac{P_2 B}{P_1}\right)^{\frac{2}{\alpha}} x^2 \int_1^{\infty}\left[1 - \left(1 - \rho^{\mathrm{e}}_2\left(1 - \frac{1}{1 + (\beta B)^{-1} T\nu^{-\alpha}}\right)\right)\left(1 - \rho_2\left(1 - \frac{1}{1 + B^{-1} T_{\mathrm{FR},1}\nu^{-\alpha}}\right)\right)\right] \mathrm{v}d\mathrm{v}\right) \end{aligned} \quad (47)$$

$$\begin{aligned} &\frac{2\pi\lambda_1}{\mathcal{A}_1} \int_0^{\infty} x \exp\left(-\frac{T}{\mathrm{SNR}_1(x,\beta)} - \frac{T_{\mathrm{FR},1}}{\mathrm{SNR}_1(x,1)}\right) \\ &\exp\left\{-\pi\lambda_1 x^2\left[1 + 2\xi\left(\rho^{\mathrm{i}}_1, \rho_1, T, T_{\mathrm{FR},1}, \alpha\right)\right] - \pi\lambda_2\left(\frac{P_2 B}{P_1}\right)^{\frac{2}{\alpha}} x^2\left[1 + 2\xi\left(\rho^{\mathrm{e}}_2, \rho_2, (\beta B)^{-1} T, B^{-1} T_{\mathrm{FR},1}, \alpha\right)\right]\right\} dx \end{aligned} \quad (48)$$

Similarly, $\mathbb{E}\left[\exp\left(-\hat{s}_2\hat{I}^{\mathrm{e}}_{x,2}-s_2I_{x,2}\right)\right]=$

$$\exp\left(-2\pi\lambda_2\int_{\left(\frac{P_2B}{P_1}\right)^{\frac{1}{\alpha}}x}^{\infty}\left[1-\left(1-\rho_2^{\mathrm{e}}\left(1-\frac{1}{1+\hat{s}_2P_2v^{-\alpha}}\right)\right)\right.\right.$$
$$\left.\left.\times\left(1-\rho_2\left(1-\frac{1}{1+s_2P_2v^{-\alpha}}\right)\right)\right]\mathrm{vdv}\right).$$

Hence, $\mathcal{L}\left(x^{\alpha}P_1^{-1}T, x^{\alpha}P_1^{-1}T_{\mathrm{FR},1}, x^{\alpha}(\beta P_1)^{-1}T, x^{\alpha}P_1^{-1}T_{\mathrm{FR},1}\right)$ is given by (47).

Deconditioning on x, the numerator of (44) can is given by (48), where $\xi\left(a,b,c,d,e\right)=$

$$\int_1^{\infty}\left[1-\left(1-a\left(1-\frac{1}{1+\mathrm{c}v^{-e}}\right)\right)\left(1-b\left(1-\frac{1}{1+\mathrm{d}v^{-e}}\right)\right)\right]\mathrm{vdv}.$$

The denominator of (44) can be obtained from (9), we have the conditional coverage probability given in Theorem 2.

Appendix C

Proof of Theorem 3 A user $u\in\mathcal{U}_1$ with SINR $<T_{\mathrm{FR},1}$ is allocated a sub-channel from available sub-channel subset $\mathbf{C}_2$ and will experience new fading power $\hat{H}_x$ and out-of-cell interference $I^{\mathrm{e}}_{x,1}+I^{i}_{x,2}$, instead of H_x and $\sum_{j=1}^{2}I_{x,j}$. The interference $I^{\mathrm{e}}_{x,1}+I^{i}_{x,2}$ is different from H_x and $\sum_{j=1}^{2}I_{x,j}$ because of the disparity of each BS' load. Then, conditioning on its previous SINR, the coverage probability of the macro edge user is

$$p^{\mathrm{e}}_{\mathrm{FFR},1}\left(T\right)=\mathbb{P}\left(\frac{P_1\hat{H}_xx^{-\alpha}}{\hat{I}^{\mathrm{e}}_{x,1}+\hat{I}^{\mathrm{i}}_{x,2}+\sigma^2}>T\,\middle|\,\frac{P_1H_xx^{-\alpha}}{\sum_{j=1}^{2}I_{x,j}+\sigma^2}<T_{\mathrm{FR},1}\right).\qquad(49)$$

□

Applying the Bayes' rule, we have

$$p^{\mathrm{e}}_{\mathrm{FFR},1}\left(T\right)=\frac{\mathbb{P}\left(\frac{P_1\hat{H}_xx^{-\alpha}}{\hat{I}^{\mathrm{e}}_{x,1}+\hat{I}^{\mathrm{i}}_{x,2}+\sigma^2}>T,\frac{P_1H_xx^{-\alpha}}{\sum_{j=1}^{2}I_{x,j}+\sigma^2}<T_{FR,1}\right)}{\mathbb{P}\left(\frac{P_1H_xx^{-\alpha}}{\sum_{j=1}^{2}I_{x,j}+\sigma^2}<T_{\mathrm{FR},1}\right)}$$
$$\overset{(a)}{=}\frac{\mathbb{E}\left[\exp\left(-x^{\alpha}P_1^{-1}T\left(\hat{I}^{\mathrm{e}}_{x,1}+\hat{I}^{\mathrm{i}}_{x,2}+\sigma^2\right)\right)\times\left(1-\exp\left(-x^{\alpha}P_1^{-1}T_{\mathrm{FR},1}\left(\sum_{j=1}^{2}I_{x,j}+\sigma^2\right)\right)\right)\right]}{1-\mathbb{E}\left[\exp\left(-x^{\alpha}P_1^{-1}T_{\mathrm{FR},1}\left(\sum_{j=1}^{2}I_{x,j}+\sigma^2\right)\right)\right]}\qquad(50)$$

where (a) follows from the independence of $\hat{H}_x$ and H_x, and $\hat{H}_x,H_x\sim\exp(1)$. The numerator of (50) can be decomposed as

$$\mathbb{E}\left[\exp\left(-x^{\alpha}P_1^{-1}T\left(\hat{I}^{\mathrm{e}}_{x,1}+\hat{I}^{\mathrm{i}}_{x,2}+\sigma^2\right)\right)\right]-\mathbb{E}\left[\exp\left(-x^{\alpha}P_1^{-1}T\left(\hat{I}^{\mathrm{e}}_{x,1}+\hat{I}^{\mathrm{i}}_{x,2}+\sigma^2\right)\right)\times\exp\left(-x^{\alpha}P_1^{-1}T_{\mathrm{FR},1}\left(\sum_{j=1}^{2}I_{x,j}+\sigma^2\right)\right)\right]$$

The first term of the numerator represents the SINR on the newly allocated sub-channel and following the method of Theorem 1 gives (51).

$$\frac{2\pi\lambda_1}{\mathcal{A}_1}\int_0^{\infty}x\exp\left(-\frac{T}{\mathrm{SNR}_1\left(x\right)}\right)\exp\left\{-\pi\lambda_1Q\left(T,\alpha,1,\rho_1^{\mathrm{e}}\right)x^2-\pi\lambda_2(P_2/P_1)^{2/\alpha}Q\left(T,\alpha,B,\rho_2^{\mathrm{i}}\right)x^2\right\}dx\qquad(51)$$

$$\exp\left(-2\pi\lambda_1x^2\int_1^{\infty}\left[1-\left(1-\rho_1^{\mathrm{e}}\left(1-\frac{1}{1+Tv^{-\alpha}}\right)\right)\left(1-\rho_1\left(1-\frac{1}{1+T_{\mathrm{FR},1}v^{-\alpha}}\right)\right)\right]\mathrm{vdv}\right)\times$$
$$\exp\left(-2\pi\lambda_2\left(\frac{P_2B}{P_1}\right)^{\frac{2}{\alpha}}x^2\int_1^{\infty}\left[1-\left(1-\rho_2^{\mathrm{i}}\left(1-\frac{1}{1+B^{-1}Tv^{-\alpha}}\right)\right)\left(1-\rho_2\left(1-\frac{1}{1+B^{-1}T_{\mathrm{FR},1}v^{-\alpha}}\right)\right)\right]\mathrm{vdv}\right)\qquad(52)$$

$$\frac{2\pi\lambda_1}{\mathcal{A}_1}\int_0^{\infty}x\exp\left(-\frac{T+T_{\mathrm{FR},1}}{\mathrm{SNR}_1}\left(x,1\right)\right)$$
$$\exp\left\{-\pi\lambda_1x^2\left[1+2\xi\left(\rho_1^{\mathrm{e}},\rho_1,T,T_{\mathrm{FR},1},\alpha\right)\right]-\pi\lambda_2\left(\frac{P_2B}{P_1}\right)^{\frac{2}{\alpha}}x^2\left[1+2\xi\left(\rho_2^{\mathrm{i}},\rho_2,B^{-1}T,B^{-1}T_{\mathrm{FR},1},\alpha\right)\right]\right\}dx\qquad(53)$$

$$\mathbb{P}\left(\tilde{N}_k^{\mathrm{i}}=n\right)=\frac{3.5^{3.5}}{n!}\frac{\Gamma\left(n+4.5\right)}{\Gamma\left(3.5\right)}\left(\frac{\lambda_u\mathcal{A}_kp_{c,k}\left(T_{\mathrm{FR},k}\right)}{\lambda_k}\right)^n\left(3.5+\frac{\lambda_u\mathcal{A}_kp_{c,k}\left(T_{\mathrm{FR},k}\right)}{\lambda_k}\right)^{-(n+4.5)}\qquad(54)$$

$$\mathbb{P}\left(\tilde{N}_k^{\mathrm{e}}=n\right)=\frac{3.5^{3.5}}{n!}\frac{\Gamma\left(n+4.5\right)}{\Gamma\left(3.5\right)}\left(\frac{\lambda_u\mathcal{A}_k\left(1-p_{c,k}\left(T_{\mathrm{FR},k}\right)\right)}{\lambda_k}\right)^n\left(3.5+\frac{\lambda_u\mathcal{A}_k\left(1-p_{c,k}\left(T_{\mathrm{FR},k}\right)\right)}{\lambda_k}\right)^{-(n+4.5)}\qquad(55)$$

Now conditioning on the second term of the numerator and following the method of Theorem 1, the joint Laplace transform $\mathcal{L}\left(\hat{s}_1, s_1, \hat{s}_2, s_2\right)$ of $\hat{I}^{\mathrm{e}}_{x,1}, I_{x,1}, \hat{I}^{\mathrm{i}}_{x,2}$, and $I_{x,2}$ evaluated at $\left(x^\alpha P_1^{-1} T, x^\alpha P_1^{-1} T_{\mathrm{FR},1}, x^\alpha P_1^{-1} T,\ x^\alpha P_1^{-1} T_{\mathrm{FR},1}\right)$ is

$$\begin{aligned}
&\mathcal{L}\left(\hat{s}_1, s_1, \hat{s}_2, s_2\right)\\
&= \mathbb{E}\left[\exp\left(-\hat{s}_1 \hat{I}^{\mathrm{e}}_{x,1} - s_1 I_{x,1} - \hat{s}_2 \hat{I}^{\mathrm{i}}_{x,2} - s_2 I_{x,2}\right)\right]\\
&= \mathbb{E}\left[\exp\left(-\hat{s}_1 \hat{I}^{\mathrm{e}}_{x,1} - s_1 I_{x,1}\right)\right] \mathbb{E}\left[\exp\left(-\hat{s}_2 \hat{I}^{\mathrm{i}}_{x,2} - s_2 I_{x,2}\right)\right]
\end{aligned}$$

Hence, $\mathcal{L}\left(x^\alpha P_1^{-1} T, x^\alpha P_1^{-1} T_{\mathrm{FR},1}, x^\alpha P_1^{-1} T, x^\alpha P_1^{-1} T_{\mathrm{FR},1}\right)$ is given by (52).

Deconditioning on x, the second term of the numerator can be expressed as (53), where $\xi\ (a, b, c, d, e)$ is originally defined in Theorem 2.

The denominator of (50) also can be obtained from (9), we have $1 - p_{c,1}\left(T_{\mathrm{FR},1}\right)$. Plugging back into (50), the conditional coverage probability given in Theorem 3 is obtained.

Appendix D

Proof of Lemma 4 Let $\tilde{N}^{\mathrm{i}}_k$ and $\tilde{N}^{\mathrm{e}}_k$ be the numbers of other interior and edge users, conditioned on the typical user being associated with that BS in the kth tier, respectively. Their PMFs are derived in a similar way in [23], given by (54) and (55). □

According to the proposed random resource allocation scheme, t^{i}_k and t^{e}_k satisfy

$$t^{\mathrm{i}}_1 = \begin{cases} 1 & \text{if } \tilde{N}^{\mathrm{i}}_1 + 1 \le |\mathbf{C}_1| \\ |\mathbf{C}_1| \big/ \tilde{N}^{\mathrm{i}}_1 + 1 & \text{otherwise} \end{cases},$$

$$t^{\mathrm{e}}_1 = \begin{cases} 1 & \text{if } \tilde{N}^{\mathrm{e}}_1 + 1 \le |\mathbf{C}_2| \\ |\mathbf{C}_2| \big/ \tilde{N}^{\mathrm{e}}_1 + 1 & \text{otherwise} \end{cases},$$

$$t^{\mathrm{i}}_2 = \begin{cases} 1 & \text{if } \tilde{N}^{\mathrm{i}}_2 + 1 \le |\mathbf{C}_2| \\ |\mathbf{C}_2| / \tilde{N}^{\mathrm{i}}_2 + 1 & \text{otherwise} \end{cases},$$

$$t^{\mathrm{e}}_2 = \begin{cases} 1 & \text{if } \tilde{N}^{\mathrm{e}}_2 + 1 \le |\mathbf{C}_1| \\ |\mathbf{C}_1| / \tilde{N}^{\mathrm{e}}_2 + 1 & \text{otherwise} \end{cases}.$$

Thus, $\mathbb{E}\left[t^{\mathrm{i}}_1\right]$ can be derived as

$$\begin{aligned}
\mathbb{E}\left[t^{\mathrm{i}}_1\right] &= \sum_{n=0}^{|\mathbf{C}_1|-1} \left(\tilde{N}^{\mathrm{i}}_1 = n\right) + \sum_{n=|\mathbf{C}_1|}^{\infty} \frac{|\mathbf{C}_1|}{n+1} \left(\tilde{N}^{\mathrm{i}}_1 = n\right)\\
&= \sum_{n=0}^{\infty} \frac{|\mathbf{C}_1|}{n+1} \left(\tilde{N}^{\mathrm{i}}_1 = n\right) + \sum_{n=0}^{|\mathbf{C}_1|-1} \left(1 - \frac{|\mathbf{C}_1|}{n+1}\right) \left(\tilde{N}^{\mathrm{i}}_1 = n\right)
\end{aligned} \tag{56}$$

Besides, the mean proportion of sub-channel allocated to the macro interior user is derived by (57), where (a) follows from the change of variables $i = n + 1$. Plugging back into (56), we have

$$\begin{aligned}
\mathbb{E}\left[t^{\mathrm{i}}_1\right] = &\frac{\lambda_1 |\mathbf{C}_1|}{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)} \left[1 - \left(1 + \frac{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)}{3.5\lambda_1}\right)^{-3.5}\right]\\
&+ \sum_{n=0}^{|\mathbf{C}_1|-1} \left(1 - \frac{|\mathbf{C}_1|}{n+1}\right) \mathbb{P}\left(\tilde{N}^{\mathrm{i}}_1 = n\right)
\end{aligned}$$

Following the same derivation procedure, the $\mathbb{E}\left[t^{\mathrm{e}}_1\right]$, $\mathbb{E}\left[t^{\mathrm{i}}_2\right]$, and $\mathbb{E}\left[t^{\mathrm{e}}_2\right]$ can be obtained.

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{\tilde{N}^{\mathrm{i}}_1 + 1}\right] &= \sum_{n=0}^{\infty} \frac{1}{n+1} \mathbb{P}\left(\tilde{N}^{\mathrm{i}}_1 = n\right)\\
&\overset{(a)}{=} \sum_{i=1}^{\infty} \frac{3.5^{3.5}}{i!} \frac{\Gamma\left(n+3.5\right)}{\Gamma\left(3.5\right)} \left(\frac{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)}{\lambda_1}\right)^{i-1}\\
&\quad \times \left(3.5 + \frac{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)}{\lambda_1}\right)^{-(i+3.5)}\\
&= \frac{\lambda_1}{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)} \sum_{i=1}^{\infty} \frac{3.5^{3.5}}{i!} \frac{\Gamma\left(n+3.5\right)}{\Gamma\left(3.5\right)}\\
&\quad \times \left(\frac{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)}{\lambda_1}\right)^{i} \left(3.5 + \frac{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)}{\lambda_1}\right)^{-(i+3.5)}\\
&= \frac{\lambda_1}{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)} \left[\sum_{i=0}^{\infty} \mathbb{P}\left(N^{\mathrm{i}}_k = i\right) - \mathbb{P}\left(N^{\mathrm{i}}_k = 0\right)\right]\\
&= \frac{\lambda_1}{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)} \left[1 - \left(1 + \frac{\lambda_u \mathcal{A}_1 p_{c,1}\left(T_{\mathrm{FR},1}\right)}{3.5\lambda_1}\right)^{-3.5}\right]
\end{aligned} \tag{57}$$

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 61271263).

Authors' contributions

LG and SC conceived the proposed scheme. SC conducted the detailed derivation to evaluate the performance of the proposed scheme and wrote the manuscript. LG and ZS reviewed the manuscript. All authors have read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

Publisher's Note

Received: 27 October 2016 Accepted: 27 September 2017
Published online: 17 October 2017

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